Orchestrating with Contracts

Technical Report

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Abstract. This technical report consists of two sections that expose proofs of correctness for the partial composition operators and the contract derivation function defined in our theoretical paper [1]. The definitions, sections, and algorithms referred in this report relate to those of the theoretical paper.

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A Algebraic Properties of Partial Contract Composition Operators

In this appendix we list the algebraic properties that the partial contract composition operators (i.e., $\triangle$, $\blacktriangle$, $\overline{\wedge}$, and $\veebar$) must satisfy. These lemmas are the theoretical foundation required to prove the correctness of the semantic derivation function. Additionally, we also provide the reader with all the proofs required to guarantee the partial contract composition operators satisfy the algebraic laws.

All the lemmas described in this chapter are defined as equality relations between two expressions that compose partial contracts.

Definition 1 (Partial Contract Expression Equality). *Two expressions that compose partial contracts are equal iff the evaluation of both expressions result in the same partial contract. Two partial contracts are* equal *iff their assertions, frame, and footprint are equal.*

According to Definition 15, Orc expressions have a precondition and a postcondition partial contract. Based on the structure of Algorithms 1 and 2, our strategy to prove each lemma consists of analyzing the different cases of communication and interference between the Orc expressions identified by the composed partial contracts. Then, for each distinct case, we apply some proof rules until the two expressions of partial contracts under comparison are syntactically equal. Table 1 lists the names of the rules, and provides some description of each of them.

Table 1. List of Proof Rules.

expand	Expand the inner-most partial contract composition operator
$\oplus$ *comm*	Commutativity of $\oplus$, where $\oplus$ is either $\wedge$, $\triangle$ or $\blacktriangle$
$\oplus$ *assoc*	Associativity of $\oplus$, where $\oplus$ is either $\wedge$, $\triangle$ or $\blacktriangle$
$\wedge$ *dist*	Distributivity of $\wedge$
dist $\wedge/\vee$	Distributivity of $\wedge$ over $\vee$
right exchange y/z	Exchange references to y by references to z in the right-most partial contract expression
subs clause1/clause2 in assertion	Substitution of *clause1* by *clause2* in an *assertion*

We assume that the preconditions and postconditions included in the partial contracts composed by $\triangle$ and $\blacktriangle$ are in conjuncted normal form. Thus, we find convenient to interprete the preconditions and postconditions of a expression C as the finite sequence of conjuncted assertion clauses i.e.,

$$P_C \triangleq \bigwedge P_{C_n} \triangleq P_{C_1} \wedge \ldots \wedge P_{C_i} \wedge \ldots \wedge P_{C_n}$$

$$Q_C \triangleq \bigwedge Q_{C_m} \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_j} \wedge \ldots \wedge Q_{C_m}$$

However, the exhaustive nature of our proof strategy could entail some readability and understanding difficulties to the reader. We aim to reduce these difficulties by taking a series of decisions.

On one hand, we do not reason about the equality of the frame and footprint of the two expressions of partial contracts under comparison. Our proofs focus in the assertions of the composed partial contracts, while denoting the frame and footprint as "...". Moreover, after expanding all the partial contract composition operators used in the two expressions under comparison, we omit all references to the frame and footprint of partial contracts.

Lets discuss the legitimacy of avoiding to reason about the resulting frame and footprint of the expressions of partial contracts under comparison. The lemmas of interest describe the commutativity, associativity, and distributivity properties of the partial contract composition operators. Independently from the operator in use, the footprint resulting from the composition of two or more partial contracts is the union of the footprints of such partial contracts. Similarly, the frame resulting from the composition of two or more partial contracts is the union of the frames of such partial contracts excluding the references to any communication message.

The union operator is commutative, associative, and distributive. Hence, we can certainly deduce that the frame and footprint resulting from the evaluation of any two expressions under comparison is always the same. This deduction also applies in the case that the Orc expressions identified by the partial contracts of any two expressions that are presumably equal communicate via message passing and are executed in the same order (e.g., associativity of $\triangle$).

On another hand, despite Orc expressions may communicate and interfere via different locations, we consider that communication and interference between Orc expressions only occur via one location. We use the literal s to refer to this location. We use the literal r to reference the values of input parameters and the pre-state values of state variables, and r' to reference the post-state values of state variables. Moreover, we use the literals y and z to refer to the intermediate state of s. These literals are fresh locations.

Finally, most lemmas compare expressions that compose three partial contracts. Each case where only two out of the related three Orc expressions communicate or interfere produces three subcases. We claim that we only need to consider one of this subcases while ensuring case analysis completeness. The following two lemmas demonstrate the legitimacy of our decision.

Lemma 1 (Precondition Case Coverage Reduction). *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be the precondition partial contracts corresponding to the Orc expressions C, D, and E, respectively. $\otimes$ is the boolean* exclusive-or *operator. Consider the following expression:*

$$\mathcal{C} \triangle \mathcal{D} \triangle \mathcal{E}$$

The evaluation of such expression when only two of the Orc expressions communicate (i.e., C with D, or C with E, or D with E) yields the same result *i.e.*,

$$\forall \mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{C},\mathcal{D},\mathcal{E} \in \mathit{PartialContract} \,.\, \mathcal{X} \neq \mathcal{Y} \wedge \mathcal{Y} \neq \mathcal{Z} \wedge \mathcal{X} = \mathcal{C} \otimes \mathcal{D} \wedge \mathcal{Y} = \mathcal{D} \otimes \mathcal{E} \wedge$$
$$\exists s \in \mathit{MemLocs} \,.\, \mathit{communicates?}(s,\mathcal{X},\mathcal{Y}) \wedge (\nexists t \in \mathit{MemLocs} \,.\, \mathit{communicates?}(t,\mathcal{X},\mathcal{Z})$$
$$\vee\, \mathit{communicates?}(t,\mathcal{Z},\mathcal{X}) \vee \mathit{communicates?}(t,\mathcal{Y},\mathcal{Z}) \vee \mathit{communicates?}(t,\mathcal{Z},\mathcal{Y})) \Rightarrow$$
$$(\mathcal{Z}\triangle\mathcal{X})\triangle\mathcal{Y} = \mathcal{Z}\triangle(\mathcal{X}\triangle\mathcal{Y}) = (\mathcal{X}\triangle\mathcal{Z})\triangle\mathcal{Y} = \mathcal{X}\triangle(\mathcal{Z}\triangle\mathcal{Y}) = (\mathcal{X}\triangle\mathcal{Y})\triangle\mathcal{Z} = \mathcal{X}\triangle(\mathcal{Y}\triangle\mathcal{Z})$$

Proof. *X communicates with Y via a location s. Let P_{Y_i} be the clause of P_Y (i.e., P_D or P_E) that includes a reference to s. According to the definition of $\triangle$,*

$$P'_Y = P_{Y_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{Y_n}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}(\mathcal{Z}\triangle\mathcal{X})\triangle\mathcal{Y} = \mathcal{Z}\triangle(\mathcal{X}\triangle\mathcal{Y}) = (\mathcal{X}\triangle\mathcal{Z})\triangle\mathcal{Y} = \\ \mathcal{X}\triangle(\mathcal{Z}\triangle\mathcal{Y}) = (\mathcal{X}\triangle\mathcal{Y})\triangle\mathcal{Z} = \mathcal{X}\triangle(\mathcal{Y}\triangle\mathcal{Z})\end{array}}{\begin{array}{c}(P_Z \wedge P_X, \ldots)\triangle\mathcal{Y} = \mathcal{Z}\triangle(P_X \wedge P'_Y, \ldots) = (P_X \wedge P_Z, \ldots)\triangle\mathcal{Y} = \\ \mathcal{X}\triangle(P_Z \wedge P_Y, \ldots) = (P_X \wedge P'_Y, \ldots)\triangle\mathcal{Z} = \mathcal{X}\triangle(P_Y \wedge P_Z, \ldots)\end{array}}\ \textsc{expand}}{\begin{array}{c}(P_Z \wedge P_X) \wedge P'_Y = P_Z \wedge (P_X \wedge P'_Y) = (P_X \wedge P_Z) \wedge P'_Y = \\ P_X \wedge (P_Z \wedge P'_Y) = (P_X \wedge P'_Y) \wedge P_Z = P_X \wedge (P'_Y \wedge P_Z)\end{array}}\ \textsc{expand}}{\begin{array}{c}(P_Z \wedge P_X) \wedge P'_Y = (P_Z \wedge P_X) \wedge P'_Y = (P_Z \wedge P_X) \wedge P'_Y = \\ (P_Z \wedge P_X) \wedge P'_Y = (P_Z \wedge P_X) \wedge P'_Y = (P_Z \wedge P_X) \wedge P'_Y\end{array}}\ \wedge\ \textsc{dist+comm}$$

Lemma 2 (Postcondition Case Coverage Reduction). *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be the postcondition partial contracts corresponding to the Orc expressions C, D, and E, respectively. $\otimes$ is the boolean* exclusive-or *operator. Consider the following expression:*

$$\mathcal{C}\blacktriangle\mathcal{D}\blacktriangle\mathcal{E}$$

The evaluation of such expression when only two of the Orc expressions communicate or interfere (i.e., C with D, or C with E, or D with E) yields the same result i.e.,

$$\forall \mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{C},\mathcal{D},\mathcal{E} \in \mathit{PartialContract} \,.\, \mathcal{X} \neq \mathcal{Y} \wedge \mathcal{Y} \neq \mathcal{Z} \wedge \mathcal{X} = \mathcal{C} \otimes \mathcal{D} \wedge \mathcal{Y} = \mathcal{D} \otimes \mathcal{E} \wedge$$
$$\exists s \in \mathit{MemLocs} \,.\, \mathit{communicates?}(s,\mathcal{X},\mathcal{Y}) \vee \mathit{interferes?}(s,\mathcal{X},\mathcal{Y}) \wedge$$
$$(\nexists t \in \mathit{MemLocs} \,.\, \mathit{communicates?}(t,\mathcal{X},\mathcal{Z}) \vee \mathit{communicates?}(t,\mathcal{Z},\mathcal{X}) \vee$$
$$\mathit{communicates?}(t,\mathcal{Y},\mathcal{Z}) \vee \mathit{communicates?}(t,\mathcal{Z},\mathcal{Y}) \vee \mathit{interferes?}(s,\mathcal{X},\mathcal{Z}) \vee$$
$$\mathit{interferes?}(s,\mathcal{Y},\mathcal{Z})) \Rightarrow$$
$$(\mathcal{Z}\blacktriangle\mathcal{X})\blacktriangle\mathcal{Y} = \mathcal{Z}\blacktriangle(\mathcal{X}\blacktriangle\mathcal{Y}) = (\mathcal{X}\blacktriangle\mathcal{Z})\blacktriangle\mathcal{Y} = \mathcal{X}\blacktriangle(\mathcal{Z}\blacktriangle\mathcal{Y}) = (\mathcal{X}\blacktriangle\mathcal{Y})\blacktriangle\mathcal{Z} = \mathcal{X}\blacktriangle(\mathcal{Y}\blacktriangle\mathcal{Z})$$

Proof. *Consider the following three cases:*

1. *X communicates with Y via a location s, and they do not interfere. Let Q_{Y_i} be the clause of Q_Y (i.e., Q_D or Q_E) that includes a reference to the pre-state of s. According to the definition of $\blacktriangle$,*

$$Q'_Y = Q_{Y_1} \wedge \ldots \wedge Q_{Y_i}[r/r'] \wedge \ldots \wedge Q_{Y_n}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}(\mathcal{Z}\blacktriangle\mathcal{X})\blacktriangle\mathcal{Y} = \mathcal{Z}\blacktriangle(\mathcal{X}\blacktriangle\mathcal{Y}) = (\mathcal{X}\blacktriangle\mathcal{Z})\blacktriangle\mathcal{Y} = \\ \mathcal{X}\blacktriangle(\mathcal{Z}\blacktriangle\mathcal{Y}) = (\mathcal{X}\blacktriangle\mathcal{Y})\blacktriangle\mathcal{Z} = \mathcal{X}\blacktriangle(\mathcal{Y}\blacktriangle\mathcal{Z})\end{array}}{\begin{array}{c}(Q_Z \wedge Q_X, \ldots)\blacktriangle\mathcal{Y} = \mathcal{Z}\blacktriangle(Q_X \wedge Q'_Y, \ldots) = (Q_X \wedge Q_Z, \ldots)\blacktriangle\mathcal{Y} = \\ \mathcal{X}\blacktriangle(Q_Z \wedge Q_Y, \ldots) = (Q_X \wedge Q'_Y, \ldots)\blacktriangle\mathcal{Z} = \mathcal{X}\blacktriangle(Q_Y \wedge Q_Z, \ldots)\end{array}}\text{ EXPAND}}{\begin{array}{c}(Q_Z \wedge Q_X) \wedge Q'_Y = Q_Z \wedge (Q_X \wedge Q'_Y) = (Q_X \wedge Q_Z) \wedge Q'_Y = \\ Q_X \wedge (Q_Z \wedge Q'_Y) = (Q_X \wedge Q'_Y) \wedge Q_Z = Q_X \wedge (Q'_Y \wedge Q_Z)\end{array}}\text{ EXPAND}}{\begin{array}{c}(Q_Z \wedge Q_X) \wedge Q'_Y = (Q_Z \wedge Q_X) \wedge Q'_Y = (Q_Z \wedge Q_X) \wedge Q'_Y = \\ (Q_Z \wedge Q_X) \wedge Q'_Y = (Q_Z \wedge Q_X) \wedge Q'_Y = (Q_Z \wedge Q_X) \wedge Q'_Y\end{array}}\ \wedge\text{ DIST+COMM}$$

2. *X is independent from Y, and they interfere via a location s. Let Q_{X_i} be the clause of Q_X (i.e., Q_C or Q_D) that includes a reference to the post-state value of s. According to the definition of $\blacktriangle$,*

$$Q'_X = Q_{X_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{X_n}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}(\mathcal{Z}\blacktriangle\mathcal{X})\blacktriangle\mathcal{Y} = \mathcal{Z}\blacktriangle(\mathcal{X}\blacktriangle\mathcal{Y}) = (\mathcal{X}\blacktriangle\mathcal{Z})\blacktriangle\mathcal{Y} = \\ \mathcal{X}\blacktriangle(\mathcal{Z}\blacktriangle\mathcal{Y}) = (\mathcal{X}\blacktriangle\mathcal{Y})\blacktriangle\mathcal{Z} = \mathcal{X}\blacktriangle(\mathcal{Y}\blacktriangle\mathcal{Z})\end{array}}{\begin{array}{c}(Q_Z \wedge Q_X, \ldots)\blacktriangle\mathcal{Y} = \mathcal{Z}\blacktriangle(Q'_X \wedge Q_Y, \ldots) = (Q_X \wedge Q_Z, \ldots)\blacktriangle\mathcal{Y} = \\ \mathcal{X}\blacktriangle(Q_Z \wedge Q_Y, \ldots) = (Q'_X \wedge Q_Y, \ldots)\blacktriangle\mathcal{Z} = \mathcal{X}\blacktriangle(Q_Y \wedge Q_Z, \ldots)\end{array}}\text{ EXPAND}}{\begin{array}{c}(Q_Z \wedge Q'_X) \wedge Q_Y = Q_Z \wedge (Q'_X \wedge Q_Y) = (Q'_X \wedge Q_Z) \wedge Q_Y = \\ Q'_X \wedge (Q_Z \wedge Q_Y) = (Q'_X \wedge Q_Y) \wedge Q_Z = Q'_X \wedge (Q_Y \wedge Q_Z)\end{array}}\text{ EXPAND}}{\begin{array}{c}(Q_Z \wedge Q'_X) \wedge Q_Y = (Q_Z \wedge Q'_X) \wedge Q_Y = (Q_Z \wedge Q'_X) \wedge Q_Y = \\ (Q_Z \wedge Q'_X) \wedge Q_Y = (Q_Z \wedge Q'_X) \wedge Q_Y = (Q_Z \wedge Q'_X) \wedge Q_Y\end{array}}\ \wedge\text{ DIST+COMM}$$

3. *X both communicates and interferes with Y via a location s. Let Q_{X_i} and Q_{Y_j} be clauses of Q_X (i.e., Q_C or Q_D) and Q_Y (i.e., Q_D or Q_E), respectively. Assume that Q_{X_i} includes a reference to the post-state value of s, and that Q_{Y_j} includes a reference to the pre-state value of s. According to the definition of $\blacktriangle$,*

$$Q'_X = Q_{X_1} \wedge \ldots \wedge Q_{X_i}[r'/y] \wedge \ldots \wedge Q_{X_n}$$
$$Q'_Y = Q_{Y_1} \wedge \ldots \wedge Q_{Y_j}[r/y] \wedge \ldots \wedge Q_{Y_m}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}(\mathcal{Z}\blacktriangle\mathcal{X})\blacktriangle\mathcal{Y} = \mathcal{Z}\blacktriangle(\mathcal{X}\blacktriangle\mathcal{Y}) = (\mathcal{X}\blacktriangle\mathcal{Z})\blacktriangle\mathcal{Y} = \\ \mathcal{X}\blacktriangle(\mathcal{Z}\blacktriangle\mathcal{Y}) = (\mathcal{X}\blacktriangle\mathcal{Y})\blacktriangle\mathcal{Z} = \mathcal{X}\blacktriangle(\mathcal{Y}\blacktriangle\mathcal{Z})\end{array}}{\begin{array}{c}(Q_Z \wedge Q_X, \ldots)\blacktriangle\mathcal{Y} = \mathcal{Z}\blacktriangle(Q'_X \wedge Q'_Y, \ldots) = (Q_X \wedge Q_Z, \ldots)\blacktriangle\mathcal{Y} = \\ \mathcal{X}\blacktriangle(Q_Z \wedge Q_Y, \ldots) = (Q'_X \wedge Q'_Y, \ldots)\blacktriangle\mathcal{Z} = \mathcal{X}\blacktriangle(Q_Y \wedge Q_Z, \ldots)\end{array}}\text{ EXPAND}}{\begin{array}{c}(Q_Z \wedge Q'_X) \wedge Q'_Y = Q_Z \wedge (Q'_X \wedge Q'_Y) = (Q'_X \wedge Q_Z) \wedge Q'_Y = \\ Q'_X \wedge (Q_Z \wedge Q'_Y) = (Q'_X \wedge Q'_Y) \wedge Q_Z = Q'_X \wedge (Q'_Y \wedge Q_Z)\end{array}}\text{ EXPAND}}{\begin{array}{c}(Q_Z \wedge Q'_X) \wedge Q'_Y = (Q_Z \wedge Q'_X) \wedge Q'_Y = (Q_Z \wedge Q'_X) \wedge Q'_Y = \\ (Q_Z \wedge Q'_X) \wedge Q'_Y = (Q_Z \wedge Q'_X) \wedge Q'_Y = (Q_Z \wedge Q'_X) \wedge Q'_Y\end{array}}\ \wedge\text{ DIST+COMM}$$

Lemmas for Partial Contract Composition Operators

Lemma 3 (Commutativity of $\triangle$). *Consider that $\mathcal{C} \triangleq (P_C, R_C, W_C)$ and $\mathcal{D} \triangleq (P_D, R_D, W_D)$ are the precondition partial contracts of Orc expressions C and D, respectively. $\triangle$ is* ***commutative****, i.e.,*

$$\mathcal{C} \triangle \mathcal{D} = \mathcal{D} \triangle \mathcal{C}$$

if C and D are independent, or in the case that any of these expressions communicates with the other, both Orc expressions have the same frame and footprint, and the precondition clauses mentioning the communication locations of both expressions are identical.

Proof. Consider the following two cases:

1. Expressions C and D are independent

$$\cfrac{\cfrac{\mathcal{C} \triangle \mathcal{D} = \mathcal{D} \triangle \mathcal{C}}{P_C \wedge P_D = P_D \wedge P_C}\ \textsc{expand}}{P_C \wedge P_D = P_C \wedge P_D}\ \wedge\ \textsc{comm}$$

2. Expressions communicate via a location s
   (a) Either C communicates with D, or D communicates with C. Assume that C communicates with D[1]. Let P_{D_i} be the clause of P_D that includes a reference to s. According to the definition of $\triangle$,

$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{D_n}$$

   We must prove that the commutativity property does not hold in this case

$$\cfrac{\mathcal{C} \triangle \mathcal{D} \neq \mathcal{D} \triangle \mathcal{C}}{P_C \wedge P'_D \neq P_D \wedge P_C}\ \textsc{expand}$$

   (b) C communicates with D, and D communicates with C. Let P_{C_i} and P_{D_j} be the clauses of P_C and P_D, respectively, that include a reference to s. According to the definition of $\triangle$,

$$P'_C \triangleq P_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{C_n}$$
$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{D_m}$$

   i. Assume that $P_{C_i} \neq P_{D_j}$. We must prove that the commutativity property does not hold in this case

$$\cfrac{\mathcal{C} \triangle \mathcal{D} \neq \mathcal{D} \triangle \mathcal{C}}{P_C \wedge P'_D \neq P_D \wedge P'_C}\ \textsc{expand}$$

[1] $C \triangle D$ and $D \triangle C$ are symmetric, and thus there is no need to analyze both cases

ii. Assume that $P_{C_i} = P_{D_j}$. We must prove that the commutativity property holds in this case.

$$P''_D \triangleq P_{D_1} \wedge \ldots \wedge P_{C_i} \wedge \ldots \wedge P_{D_m}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{C} \triangle \mathcal{D} = \mathcal{D} \triangle \mathcal{C}}{P_C \wedge P'_D = P_D \wedge P'_C}\ \text{EXPAND}}{P_C \wedge P'_D = P''_D \wedge P'_C}\ \text{SUBS } P_{D_j}/P_{C_i} \text{ IN } P_D}{P_C \wedge P'_D = P'_D \wedge P_C}\ \wedge\ \text{ASSOC}}{P_C \wedge P'_D = P_C \wedge P'_D}\ \wedge\ \text{COMM}$$

Lemma 4 (Commutativity of ▲). *Consider that $\mathcal{C} \triangleq (Q_C, R_C, W_C)$ and $\mathcal{D} \triangleq (Q_D, R_D, W_D)$ are the postcondition partial contracts of Orc expressions C and D, respectively. ▲ is* ***commutative*** *i.e.,*

$$\mathcal{C} \blacktriangle \mathcal{D} = \mathcal{D} \blacktriangle \mathcal{C}$$

if C and D are independent and non-interfering, or in the case that any of these expressions interferes with the other, both Orc expressions have the same frame and footprint, and the postcondition clauses mentioning any communication location of both expressions are identical.

Proof. *Consider the following three cases:*

1. *C and D are independent and non-interfering*

$$\dfrac{\dfrac{\mathcal{C} \blacktriangle \mathcal{D} = \mathcal{D} \blacktriangle \mathcal{C}}{Q_C \wedge Q_D = Q_D \wedge Q_C}\ \text{EXPAND}}{Q_C \wedge Q_D = Q_C \wedge Q_D}\ \wedge\ \text{COMM}$$

2. *Expressions communicate via a location s and do not interfere*
   (a) *Either C communicates with D, or D communicates with C. Assume that C communicates with D*[2]*. Let Q_{D_i} be the clause of Q_D that references the pre-state value of s. According to the definition of ▲,*

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$

We must prove that the commutativity property does not hold in this case

$$\dfrac{\mathcal{C} \blacktriangle \mathcal{D} \neq \mathcal{D} \blacktriangle \mathcal{C}}{Q_C \wedge Q'_D \neq Q_D \wedge Q_C}\ \text{EXPAND}$$

[2] $C \blacktriangle D$ and $D \blacktriangle C$ are symmetric, and thus there is no need to analyze both cases

(b) C communicates with D, and D communicates with C. Under the actual set of assumptions, C and D interfere via s. Case 4a captures this subcase.

3. Expressions do not communicate and interfere via a location s. Let Q_{C_i} and Q_{D_j} be the clauses of Q_C and Q_D, respectively, that include a reference to the post-state value of s. According to the definition of $\blacktriangle$,

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{D_m}$$

(a) Assume that $Q_{C_i} \neq Q_{D_j}$. We must prove that the commutativity property does not hold in this case

$$\frac{\mathcal{C}\blacktriangle\mathcal{D} \neq \mathcal{D}\blacktriangle\mathcal{C}}{Q'_C \wedge Q_D \neq Q'_D \wedge Q_C}\ \text{EXPAND}$$

(b) Assume that $Q_{C_i} = Q_{D_j}$. We must prove that the commutativity property holds in this case

$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{C_i} \wedge \ldots \wedge Q_{D_m}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\mathcal{C}\blacktriangle\mathcal{D} = \mathcal{D}\blacktriangle\mathcal{C}}{Q'_C \wedge Q_D = Q'_D \wedge Q_C}\ \text{EXPAND}}{Q'_C \wedge Q''_D = Q'_D \wedge Q_C}\ \text{SUBS } Q_{D_j}/Q_{C_i} \text{ IN } Q_D}{Q_C \wedge Q'_D = Q'_D \wedge Q_C}\ \wedge\ \text{ASSOC}}{Q_C \wedge Q'_D = Q_C \wedge Q'_D}\ \wedge\ \text{COMM}$$

4. Expressions communicate and interfere via a location s

(a) C communicates with D, and D communicates with C. Under the actual set of assumptions, C and D interfere via s. Let Q_{C_i} and Q_{D_j} be the clauses of Q_C and Q_D, respectively, that include references to the pre-state and post-state values of s. According to the definition of $\blacktriangle$,

i. Assume that $Q_{C_i} \neq Q_{D_j}$. We must prove that the commutativity property does not hold in this case

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r'/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r/y] \wedge \ldots \wedge Q_{C_n}$$

$$\frac{\mathcal{C}\blacktriangle\mathcal{D} \neq \mathcal{D}\blacktriangle\mathcal{C}}{Q'_C \wedge Q'_D \neq Q''_D \wedge Q''_C}\ \text{EXPAND}$$

ii. Assume that $Q_{C_i} = Q_{D_j}$. We must prove that the commutativity property holds

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[s'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{C_i} \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{C_i}[s/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q'''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{C_i}[s'/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[s/y] \wedge \ldots \wedge Q_{C_n}$$

$$\cfrac{\mathcal{C} \blacktriangle \mathcal{D} = \mathcal{D} \blacktriangle \mathcal{C}}{\cfrac{(Q_C, \ldots) \blacktriangle (Q'_D, \ldots) = (Q'_D, \ldots) \blacktriangle (Q_C, \ldots)}{\cfrac{Q'_C \wedge Q''_D = Q'''_D \wedge Q''_C}{\cfrac{Q'_C \wedge Q''_D = Q''_D \wedge Q'_C}{Q'_C \wedge Q''_D = Q'_C \wedge Q''_D}\ \wedge\ \textsc{comm}}\ \wedge\ \textsc{assoc}}\ \textsc{expand}}\ \textsc{subs}\ Q_{D_j}/Q_{C_i}\ \textsc{in}\ Q_D$$

(b) *Either C communicates with D, or D communicates with C, and C and D interfere. Both cases are implicit subcases of 4a*

Lemma 5 (Associativity of $\triangle$). *Let $\mathcal{C} \triangleq (P_C, R_C, W_C)$, $\mathcal{D} \triangleq (P_D, R_D, W_D)$, and $\mathcal{E} \triangleq (P_E, R_E, W_E)$ be the precondition partial contracts of Orc expressions C, D, and D, respectively. $\triangle$ is* ***associative****, i.e.,*

$$(\mathcal{C} \triangle \mathcal{D}) \triangle \mathcal{E} = \mathcal{C} \triangle (\mathcal{D} \triangle \mathcal{E})$$

Proof. Consider the following two cases:

1. *Expressions C, D, and E are independent*

$$\cfrac{(\mathcal{C} \triangle \mathcal{D}) \triangle \mathcal{E} = \mathcal{C} \triangle (\mathcal{D} \triangle \mathcal{E})}{\cfrac{(P_C \wedge P_D, \ldots) \triangle \mathcal{E} = \mathcal{C} \triangle (P_D \wedge P_E, \ldots)}{\cfrac{(P_C \wedge P_D) \wedge P_E = P_C \wedge (P_D \wedge P_E)}{(P_C \wedge P_D) \wedge P_E = (P_C \wedge P_D) \wedge P_E}\ \wedge\ \textsc{comm}}\ \textsc{expand}}\ \textsc{expand}$$

2. *Expressions communicate via a location s*
   (a) *Either C communicates with D or E, or D communicates with E. Without loss of generality, according to lemma 1 we just need to capture one of these subcases. Assume that C communicates with D. Let P_{D_i} be the clause of P_D that includes a reference to s. According to the definition of $\triangle$,*

$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge true \wedge \ldots \wedge P_{D_n}$$

$$\cfrac{(\mathcal{C} \triangle \mathcal{D}) \triangle \mathcal{E} = \mathcal{C} \triangle (\mathcal{D} \triangle \mathcal{E})}{\cfrac{(P_C \wedge P'_D, \ldots) \triangle \mathcal{E} = \mathcal{C} \triangle (P_D \wedge P_E, \ldots)}{\cfrac{(P_C \wedge P'_D) \wedge P_E = P_C \wedge (P'_D \wedge P_E)}{(P_C \wedge P'_D) \wedge P_E = (P_C \wedge P'_D) \wedge P_E}\ \wedge\ \textsc{comm}}\ \textsc{expand}}\ \textsc{expand}$$

(b) *C communicates both with D and E. Let P_{D_i} and P_{E_j} be the clauses of D and E, respectively, that include a reference to s. According to the definition of $\triangle$,*

$$
\begin{aligned}
P'_D &\triangleq P_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{D_n} \\
P'_E &\triangleq P_{E_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E_m}
\end{aligned}
$$

$$
\begin{array}{c}
(\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} = \mathcal{C}\triangle(\mathcal{D}\triangle\mathcal{E}) \\ \hline
(P_C \wedge P'_D, \ldots)\triangle\mathcal{E} = \mathcal{C}\triangle(P_D \wedge P'_E, \ldots) \quad \text{EXPAND} \\ \hline
(P_C \wedge P'_D) \wedge P'_E = P_C \wedge (P'_D \wedge P'_E) \quad \text{EXPAND} \\ \hline
(P_C \wedge P'_D) \wedge P'_E = (P_C \wedge P'_D) \wedge P'_E \quad \wedge\ \text{COMM}
\end{array}
$$

Lemma 6 (Associativity of ▲). *Let $\mathcal{C} \triangleq (Q_C, R_C, W_C)$, $\mathcal{D} \triangleq (Q_D, R_D, W_D)$, and $\mathcal{E} \triangleq (Q_E, R_E, W_E)$ be the postcondition partial contracts of Orc expressions C, D, and D, respectively. ▲ is* ***associative****, i.e.,*

$$(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})$$

Proof. Consider the following four cases:

1. *Expressions C, D, and E are independent and non-interfering*

$$
\begin{array}{c}
(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}) \\ \hline
(Q_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q_D \wedge Q_E, \ldots) \quad \text{EXPAND} \\ \hline
(Q_C \wedge Q_D) \wedge Q_E = Q_C \wedge (Q_D \wedge Q_E) \quad \text{EXPAND} \\ \hline
(Q_C \wedge Q_D) \wedge Q_E = (Q_C \wedge Q_D) \wedge Q_E \quad \wedge\ \text{COMM}
\end{array}
$$

2. *Expressions communicate via a location s and do not interfere*

   (a) *Either C communicates with D, or C communicates with E, or D communicates with E. Without loss of generality, according to lemma 2 we just need to capture one of these subcases. Assume that C communicates with D. Let Q_{D_i} be the clause of Q_D that includes a reference to the pre-state value of s. According to the definition of ▲,*

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$

$$
\begin{array}{c}
(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}) \\ \hline
(Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q_D \wedge Q_E, \ldots) \quad \text{EXPAND} \\ \hline
(Q_C \wedge Q'_D) \wedge Q_E = Q_C \wedge (Q'_D \wedge Q_E) \quad \text{EXPAND} \\ \hline
((Q_C \wedge Q'_D) \wedge Q_E = (Q_C \wedge Q'_D) \wedge Q_E \quad \wedge\ \text{COMM}
\end{array}
$$

(b) *C communicates both with D and E. Let Q_{D_i} and Q_{E_j} be the clauses of Q_D and Q_E, respectively, that include a reference to the pre-state value of s. According to the definition of* $\blacktriangle$,

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_j}[r/r'] \wedge \ldots \wedge Q_{E_m}$$

$$\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}{(Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q_D \wedge Q_E, \ldots)}\ \text{EXPAND}}{(Q_C \wedge Q'_D) \wedge Q'_E = Q_C \wedge (Q'_D \wedge Q'_E)}\ \text{EXPAND}}{(Q_C \wedge Q'_D) \wedge Q'_E = (Q_C \wedge Q'_D) \wedge Q'_E}\ \wedge\ \text{COMM}$$

3. *Expressions do not communicate and interfere via a location s*

(a) *Either C and D interfere, or C and E interfere, or D and E interfere. Without loss of generality, according to lemma 2 we just need to capture one of these subcases. Assume that C and D interfere. Let Q_{C_i} be the clause of C that includes a reference to the pre-state value of s. According to the definition of* $\blacktriangle$,

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$

$$\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}{(Q'_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q_D \wedge Q_E, \ldots)}\ \text{EXPAND}}{(Q'_C \wedge Q_D) \wedge Q_E = Q'_C \wedge (Q_D \wedge Q_E)}\ \text{EXPAND}}{(Q'_C \wedge Q_D) \wedge Q_E = (Q'_C \wedge Q_D) \wedge Q_E}\ \wedge\ \text{COMM}$$

(b) *C, D and E are interfering. Let Q_{C_i} and Q_{D_j} be the clauses of Q_C and Q_D that include a reference to the pre-state value of s. According to the definition of* $\blacktriangle$,

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{D_m}$$

$$\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}{(Q'_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q'_D \wedge Q_E, \ldots)}\ \text{EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q_E = Q'_C \wedge (Q'_D \wedge Q_E)}\ \text{EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q_E = (Q'_C \wedge Q'_D) \wedge Q_E}\ \wedge\ \text{COMM}$$

4. *Expressions communicate and interfere via a location s*

(a) *Only two expressions communicate. Assume that C communicates with D. Let Q_{C_i} and Q_{D_j} be clauses of Q_C and Q_D, respectively. Assume that Q_{C_i} includes a reference to the post-state value of s, and Q_{D_j} includes a reference to the pre-state value of s*

i. *C and D interfere. According to the definition of* $\blacktriangle$,

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}{(Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q_D \wedge Q_E, \ldots)}\text{ EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q_E = Q'_C \wedge (Q'_D \wedge Q_E)}\text{ EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q_E = (Q'_C \wedge Q'_D) \wedge Q_E}\ \wedge\text{ COMM}$$

ii. *Either C or D interfere with E. Assume that C and E interfere. According to the definition of* $\blacktriangle$,

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[s/s'] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{D_m}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}{(Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q_D \wedge Q_E, \ldots)}\text{ EXPAND}}{(Q'_C \wedge Q''_D) \wedge Q_E = Q'_C \wedge (Q''_D \wedge Q_E)}\text{ EXPAND}}{(Q'_C \wedge Q''_D) \wedge Q_E = (Q'_C \wedge Q''_D) \wedge Q_E}\ \wedge\text{ COMM}$$

iii. *C, D, and E interfere. According to the definition of* $\blacktriangle$,

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[s'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q''_C \triangleq Q_{C_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[s/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{D_m}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}{(Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q''_D \wedge Q_E, \ldots)}\text{ EXPAND}}{(Q''_C \wedge Q''_D) \wedge Q_E = Q''_C \wedge (Q''_D \wedge Q_E)}\text{ EXPAND}}{(Q''_C \wedge Q''_D) \wedge Q_E = (Q''_C \wedge Q''_D) \wedge Q_E}\ \wedge\text{ DIST}$$

(b) *C communicates both with D and E. Let* Q_{C_i}, Q_{D_j}, *and* Q_{E_k} *be clauses of* Q_C, Q_D, *and* Q_E, *respectively. Assume that* Q_{C_i} *includes a reference to the post-state value of s, and* Q_{D_j} *and* Q_{E_k} *include a reference to the pre-state value of s*

i. *C and D interfere. According to the definition of* $\blacktriangle$,

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[r/r'] \wedge \ldots \wedge Q_{E_o}$$

$$\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}{(Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q_D \wedge Q'_E, \ldots)}\ \text{EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q'_E = Q'_C \wedge (Q'_D \wedge Q'_E)}\ \text{EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q'_E = (Q'_C \wedge Q'_D) \wedge Q'_E}\ \wedge\ \text{COMM}$$

ii. *Either C or D interfere with E. Assume that C and E interfere. According to the definition of $\blacktriangle$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[s'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[s/s'] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q'_{D_j}[s'/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[s/y] \wedge \ldots \wedge Q_{E_o}$$

$$\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}{(Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q_D \wedge Q_E, \ldots)}\ \text{EXPAND}}{(Q'_C \wedge Q''_D) \wedge Q'_E = Q'_C \wedge (Q''_D \wedge Q'_E)}\ \text{EXPAND}}{(Q'_C \wedge Q''_D) \wedge Q'_E = (Q'_C \wedge Q''_D) \wedge Q'_E}\ \wedge\ \text{COMM}$$

iii. *C, D, and E interfere. Under the actual set of assumptions, D communicates with E. According to the definition of $\blacktriangle$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[s'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q''_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[s'/z] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[s/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[s'/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q'''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q'_{D_j}[s'/z] \wedge \ldots \wedge Q_{D_m}$$
$$Q''''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q''_{D_j}[s/z] \wedge \ldots \wedge Q_{D_m}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[s/y] \wedge \ldots \wedge Q_{E_o}$$
$$Q''_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[s/z] \wedge \ldots \wedge Q_{E_o}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}{(Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = \mathcal{C}\blacktriangle(Q''_D \wedge Q'_E, \ldots)}\ \text{EXPAND}}{(Q'_C \wedge Q'''_D) \wedge Q''_E = Q''_C \wedge (Q''''_D \wedge Q'_E)}\ \text{EXPAND}}{(Q'_C \wedge Q'''_D) \wedge Q''_E = Q'_C \wedge (Q'''_D \wedge Q''_E)}\ \text{RIGHT EXCHANGE } y/z.}{(Q'_C \wedge Q'''_D) \wedge Q''_E = (Q'_C \wedge Q'''_D) \wedge Q'_E}\ \wedge\ \text{COMM}$$

Lemma 7 (Right Distributivity of $\triangle$). *Consider that $\mathcal{C} \triangleq (P_C, R_C, W_C)$, $\mathcal{D} \triangleq (P_D, R_D, W_D)$, and $\mathcal{E} \triangleq (P_E, R_E, W_E)$ are the precondition partial contracts* *of Orc expressions* C*, * D*, and* D*, respectively. Let* $\mathcal{E}_1 \triangleq (P_{E1}, R_E, W_E)$ *and* $\mathcal{E}_2 \triangleq (P_{E2}, R_E, W_E)$ *are "subsets" of* $\mathcal{E}$*, in the sense that*

$$clauses(P_{E1}) \subseteq clauses(P_E)$$
$$clauses(P_{E2}) \subseteq clauses(P_E)$$

$\triangle$ *is* ***right distributive****, i.e.,*

$$(\mathcal{C} \triangle \mathcal{D}) \triangle \mathcal{E} = (\mathcal{C} \triangle \mathcal{E}_1) \triangle (\mathcal{D} \triangle \mathcal{E}_2)$$

if D *and* E *are independent, and either* P_{E1}*,* P_{E2}*, or both are equal to* P_E*. Furthermore, both* P_{E1} *and* P_{E2} *must include all the original clauses of* P_E *that constraint the values of the locations that* C *and* D *use to communicate with* E*.*

Proof. *Consider the following two cases:*

1. *Expressions* C*,* D*, and* E *are independent*

$$\cfrac{\cfrac{\cfrac{(\mathcal{C} \triangle \mathcal{D}) \triangle \mathcal{E} = (\mathcal{C} \triangle \mathcal{E}_1) \triangle (\mathcal{D} \triangle \mathcal{E}_2)}{(P_C \wedge P_D, \ldots) \triangle \mathcal{E} = (P_C \wedge P_{E1}, \ldots) \triangle (P_D \wedge P_{E2}, \ldots)}\ \text{EXPAND}}{(P_C \wedge P_D) \wedge P_E = (P_C \wedge P_{E1}) \wedge (P_D \wedge P_{E2})}\ \text{EXPAND}}{(P_C \wedge P_D) \wedge P_E = (P_C \wedge P_D) \wedge P_E}\ \wedge\ \text{DIST}$$

   The reader may notice that in the case that neither P_{E1} *nor* P_{E2} *are equal to* P_E*, the equality relation does not hold. Hence, for simplicity reasons, from now on in this document on we assume that either* P_{E1} *or* P_{E2} *are equal to* P_E*.*

2. *Expressions communicate via a location* s

   (a) C *communicates with* D*. Let* P_{D_i} *be the clause of* P_D *that includes a references to* s*. According to the definition of* $\triangle$*,*

$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge true \wedge \ldots \wedge P_{D_n}$$

$$\cfrac{\cfrac{\cfrac{(\mathcal{C} \triangle \mathcal{D}) \triangle \mathcal{E} = (\mathcal{C} \triangle \mathcal{E}_1) \triangle (\mathcal{D} \triangle \mathcal{E}_2)}{(P_C \wedge P'_D, \ldots) \triangle \mathcal{E} = (P_C \wedge P_{E1}, \ldots) \triangle (P_D \wedge P_{E2}, \ldots)}\ \text{EXPAND}}{(P_C \wedge P'_D) \wedge P_E = (P_C \wedge P_{E1}) \wedge (P'_D \wedge P_{E2})}\ \text{EXPAND}}{(P_C \wedge P'_D) \wedge P_E = (P_C \wedge P'_D) \wedge P_E}\ \wedge\ \text{DIST}$$

   (b) C *communicates with* E*. Let* P_{E_i}*,* P_{E1_j}*, and* P_{E2_k} *be the clauses of* P_E*,* P_{E1}*, and* P_{E2}*, respectively, that include a reference to* s*. According to the definition of* $\triangle$*,*

$$P'_E \triangleq P_{E_1} \wedge \ldots \wedge true \wedge \ldots \wedge P_{E_n}$$
$$P'_{E1} \triangleq P_{E1_1} \wedge \ldots \wedge true \wedge \ldots \wedge P_{E1_m}$$
$$P'_{E2} \triangleq P_{E2_1} \wedge \ldots \wedge true \wedge \ldots \wedge P_{E2_o}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{E}_1)\triangle(\mathcal{D}\triangle\mathcal{E}_2)}{(P_C \wedge P_D, \ldots)\triangle\mathcal{E} = (P_C \wedge P'_{E1}, \ldots)\triangle(P_D \wedge P_{E2}, \ldots)}\ \text{EXPAND}}{(P_C \wedge P_D) \wedge P'_E = (P_C \wedge P'_{E1}) \wedge (P_D \wedge P'_{E2})}\ \text{EXPAND}}{(P_C \wedge P_D) \wedge P'_E = (P_C \wedge P_D) \wedge P'_E}\ \wedge\ \text{DIST}$$

(c) *D communicates with E. Let P_{D_i}, P_{E1_j}, and P_{E2_k} be the clauses of P_D, P_{E1}, and P_{E2}, respectively, that include a reference to s. According to the definition of $\triangle$,*

$$P'_E \triangleq P_{E_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E_n}$$
$$P'_{E1} \triangleq P_{E1_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E1_m}$$
$$P'_{E2} \triangleq P_{E2_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E2_o}$$

*We must prove that $\triangle$ is **not** right distributive in this case*

$$\dfrac{\dfrac{(\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} \neq (\mathcal{C}\triangle\mathcal{E}_1)\triangle(\mathcal{D}\triangle\mathcal{E}_2)}{(P_C \wedge P_D, \ldots)\triangle\mathcal{E} \neq (P_C \wedge P_{E1}, \ldots)\triangle(P_D \wedge P'_{E2}, \ldots)}\ \text{EXPAND}}{(P_C \wedge P_D) \wedge P'_E \neq (P_C \wedge P_{E1}) \wedge (P_D \wedge P'_{E2})}\ \text{EXPAND}$$

(d) *C communicates both with D and E. Let P_{D_i}, P_{E_j}, P_{E1_k}, and P_{E2_h} be the clauses of P_D, P_E, P_{E1}, and P_{E2}, respectively, that include a reference to s. According to the definition of $\triangle$,*

$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{D_n}$$
$$P'_E \triangleq P_{E_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E_m}$$
$$P'_{E1} \triangleq P_{E1_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E1_o}$$
$$P'_{E2} \triangleq P_{E2_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E2_p}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{E}_1)\triangle(\mathcal{D}\triangle\mathcal{E}_2)}{(P_C \wedge P'_D, \ldots)\triangle\mathcal{E} = (P_C \wedge P'_{E1}, \ldots)\triangle(P_D \wedge P'_{E2}, \ldots)}\ \text{EXPAND}}{(P_C \wedge P'_D) \wedge P'_E = (P_C \wedge P'_{E1}) \wedge (P'_D \wedge P'_{E2})}\ \text{EXPAND}}{(P_C \wedge P'_D) \wedge P'_E = (P_C \wedge P'_D) \wedge P'_E}\ \wedge\ \text{DIST}$$

Lemma 8 (Right Distributivity of $\blacktriangle$). *Consider that $\mathcal{C} \triangleq (Q_C, R_C, W_C)$, $\mathcal{D} \triangleq (Q_D, R_D, W_D)$, and $\mathcal{E} \triangleq (Q_E, R_E, W_E)$ are the postcondition partial contracts of Orc expressions C, D, and D, respectively. Let $\mathcal{E}_1 \triangleq (Q_{E1}, R_E, W_E)$ and $\mathcal{E}_2 \triangleq (Q_{E2}, R_E, W_E)$ are "subsets" of $\mathcal{E}$, in the sense that*

$$\mathit{clauses}(Q_{E1}) \subseteq \mathit{clauses}(Q_E)$$
$$\mathit{clauses}(Q_{E2}) \subseteq \mathit{clauses}(Q_E)$$

*$\blacktriangle$ is **right distributive**, i.e.,*

$$(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)$$

if D and E are independent, and either Q_{E1}, Q_{E2}, or both are equal to Q_E. Furthermore, both Q_{E1} and Q_{E2} must include all the original clauses of Q_E that constraint the values of the locations that C and D use to communicate and interfere with E.

Proof. Consider the following four cases:

1. *Expressions C, D, and E are independent*

$$\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)}{(Q_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} = (Q_C \wedge Q_{E1}, \ldots)\blacktriangle(Q_D \wedge Q_{E2}, \ldots)}\ \textsc{expand}}{(Q_C \wedge Q_D) \wedge Q_E = (Q_C \wedge Q_{E1}) \wedge (Q_D \wedge Q_{E2})}\ \textsc{expand}}{(Q_C \wedge Q_D) \wedge Q_E = (Q_C \wedge Q_D) \wedge Q_E}\ \wedge\ \textsc{dist}$$

   Similarly to Lemma 7, in the case that neither Q_{E1} nor Q_{E2} are equal to Q_E, the equality relation does not hold. Hence, for simplicity reasons, from now on in this document on we assume that either Q_{E1} or Q_{E2} are equal to Q_E.

2. *Expressions communicate via a location s and do not interfere*
   (a) *C communicates with D. Let Q_{D_i} be the clause of Q_D that includes a reference to the pre-state value of s. According to the definition of $\blacktriangle$,*

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$

$$\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)}{(Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = (Q_C \wedge Q_{E1}, \ldots)\blacktriangle(Q_D \wedge Q_{E2}, \ldots)}\ \textsc{expand}}{(Q_C \wedge Q'_D) \wedge Q_E = (Q_C \wedge Q_{E1}) \wedge (Q'_D \wedge Q_{E2})}\ \textsc{expand}$$

   (b) *C communicates with E. Let Q_{E_i}, Q_{E2_j}, and Q_{E2_k} be the clause of Q_E, Q_{E1}, and Q_{E2}, respectively, that include a reference to the pre-state value of s. According to the definition of $\blacktriangle$,*

$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_i}[r/r'] \wedge \ldots \wedge Q_{E_n}$$
$$Q_{E1} \triangleq Q_{E1_1} \wedge \ldots \wedge Q_{E1_j}[r/r'] \wedge \ldots \wedge Q_{E1_m}$$
$$Q_{E2} \triangleq Q_{E2_1} \wedge \ldots \wedge Q_{E2_k}[r/r'] \wedge \ldots \wedge Q_{E2_o}$$

$$\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)}{(Q_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} = (Q_C \wedge Q'_{E1}, \ldots)\blacktriangle(Q_D \wedge Q_{E2}, \ldots)}\ \textsc{expand}}{(Q_C \wedge Q_D) \wedge Q'_E = (Q_C \wedge Q'_{E1}) \wedge (Q_D \wedge Q'_{E2})}\ \textsc{expand}}{(Q_C \wedge Q_D) \wedge Q'_E = (Q_C \wedge Q_D) \wedge Q'_E}\ \wedge\ \textsc{dist}$$

   (c) *D communicates with E. Let Q_{E_i}, Q_{E2_j}, and Q_{E2_k} be the clause of Q_E, Q_{E1}, and Q_{E2}, respectively, that include a reference to the pre-state value of s. According to the definition of $\blacktriangle$,*

$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_i}[r/r'] \wedge \ldots \wedge Q_{E_n}$$
$$Q_{E1} \triangleq Q_{E1_1} \wedge \ldots \wedge Q_{E1_j}[r/r'] \wedge \ldots \wedge Q_{E1_m}$$
$$Q_{E2} \triangleq Q_{E2_1} \wedge \ldots \wedge Q_{E2_k}[r/r'] \wedge \ldots \wedge Q_{E2_o}$$

*We must prove that ▲ is **not** right distributive in this case*

$$\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \neq (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)}{(Q_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} \neq (Q_C \wedge Q_{E1}, \ldots)\blacktriangle(Q_D \wedge Q'_{E2}, \ldots)}\ \text{EXPAND}}{(Q_C \wedge Q_D) \wedge Q'_E \neq (Q_C \wedge Q_{E1}) \wedge (Q_D \wedge Q'_{E2})}\ \text{EXPAND}$$

(d) *C communicates both with D and E. Let Q_{D_i} Q_{E_j}, Q_{E1_k}, and Q_{E2_h} be the clause of Q_D, Q_E, Q_{E1}, and Q_{E2}, respectively, that include a reference to the pre-state value of s. According to the definition of ▲,*

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$

$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_j}[r/r'] \wedge \ldots \wedge Q_{E_m}$$

$$Q_{E1} \triangleq Q_{E1_1} \wedge \ldots \wedge Q_{E1_k}[r/r'] \wedge \ldots \wedge Q_{E1_o}$$

$$Q_{E2} \triangleq Q_{E2_1} \wedge \ldots \wedge Q_{E2_h}[r/r'] \wedge \ldots \wedge Q_{E2_p}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)}{(Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = (Q_C \wedge Q'_{E1}, \ldots)\blacktriangle(Q_D \wedge Q_{E2}, \ldots)}\ \text{EXPAND}}{(Q_C \wedge Q'_D) \wedge Q'_E = (Q_C \wedge Q'_{E1}) \wedge (Q'_D \wedge Q'_{E2})}\ \text{EXPAND}}{(Q_C \wedge Q'_D) \wedge Q'_E = (Q_C \wedge Q'_D) \wedge Q'_E}\ \wedge\ \text{DIST}$$

3. *Expressions do not communicate and interfere via a location s*

(a) *Either C interferes with D or E, or D interferes with E. Without loss of generality, according to lemma 2 we just need to capture one of these subcases. Assume that C and D interfere. Let Q_{C_i} be the clause of Q_C that includes a reference to the pre-state value of s. According to the definition of ▲,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)}{(Q'_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} = (Q_C \wedge Q_{E1}, \ldots)\blacktriangle(Q_D \wedge Q_{E2}, \ldots)}\ \text{EXPAND}}{(Q'_C \wedge Q_D) \wedge Q_E = (Q'_C \wedge Q_{E1}) \wedge (Q_D \wedge Q_{E2})}\ \text{EXPAND}}{(Q'_C \wedge Q_D) \wedge Q_E = (Q'_C \wedge Q_D) \wedge Q_E}\ \wedge\ \text{DIST}$$

(b) *C, D, and E interfere. Let Q_{C_i} and Q_{D_j} be the clauses of Q_C and Q_D, respectively, that include a reference to the pre-state value of s. According to the definition of ▲,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{D_m}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)}{(Q'_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} = (Q'_C \wedge Q_{E1}, \ldots)\blacktriangle(Q'_D \wedge Q_{E2}, \ldots)}\ \text{EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q_E = (Q'_C \wedge Q_{E1}) \wedge (Q'_D \wedge Q_{E2})}\ \text{EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q_E = (Q'_C \wedge Q'_D) \wedge Q_E}\ \wedge\ \text{DIST}$$

4. *Expressions communicate and interfere via a location* s
   (a) C *communicates with* D*. Let* Q_{C_i} *and* Q_{D_j} *be clauses of* Q_C *and* Q_D*, respectively. Assume that* Q_{C_i} *includes a reference to the post-state value of* s *and* Q_{D_j} *includes a reference to the pre-state value of* s
      i. *Either* C *interferes with* D *or* E*, or* D *interferes with* E*. According to the definition of* $\blacktriangle$*,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)}{(Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = (Q_C \wedge Q_{E1}, \ldots)\blacktriangle(Q_D \wedge Q_{E2}, \ldots)}\ \text{EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q_E = (Q'_C \wedge Q_{E1}) \wedge (Q'_D \wedge Q_{E2})}\ \text{EXPAND}}{(Q'_C \wedge Q'_D) \wedge Q_E = (Q'_C \wedge Q'_D) \wedge Q_E}\ \wedge\ \text{DIST}$$

      ii. *Either* C*,* D*, and* E *interfere. According to the definition of* $\blacktriangle$*,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q''_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{D_m}$$

$$\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2)}{(Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = (Q''_C \wedge Q_{E1}, \ldots)\blacktriangle(Q''_D \wedge Q_{E2}, \ldots)}\ \text{EXPAND}}{(Q'_C \wedge Q''_D) \wedge Q_E = (Q''_C \wedge Q_{E1}) \wedge (Q''_D \wedge Q_{E2})}\ \text{EXPAND}}{(Q'_C \wedge Q''_D) \wedge Q_E = (Q'_C \wedge Q''_D) \wedge Q_E}\ \wedge\ \text{DIST}$$

   (b) C *communicates with* E*. Let* Q_{C_i}*,* Q_{E_j}*,* Q_{E1_k}*, and* Q_{E2_h} *be clauses of* Q_C*,* Q_E*,* Q_{E1}*, and* Q_{E2}*, respectively. Assume that* Q_C *includes a reference to the post-state value of* s*, and that* Q_E*,* Q_{E1}*, and* Q_{E2} *include a reference to the pre-state value of* s
      i. C *and* D *interfere. Under the actual set of assumptions,* D *and* E *communicate. According to case 2c, the property does not hold if* D *and* E *communicate*
      ii. *Either* C *or* D *interferes with* E*. Assume that* C *and* E *interfere. According to the definition of* $\blacktriangle$*,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_j}[r/y] \wedge \ldots \wedge Q_{E_m}$$
$$Q'_{E1} \triangleq Q_{E1_1} \wedge \ldots \wedge Q_{E1_k}[r/y] \wedge \ldots \wedge Q_{E1_o}$$
$$Q'_{E2} \triangleq Q_{E2_1} \wedge \ldots \wedge Q_{E2_h}[r/y] \wedge \ldots \wedge Q_{E2_p}$$

$$
\begin{array}{c}
(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2) \\
\hline
(Q_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} = (Q'_C \wedge Q_{E1}, \ldots)\blacktriangle(Q_D \wedge Q_{E2}, \ldots) \\
\hline
(Q'_C \wedge Q_D) \wedge Q'_E = (Q'_C \wedge Q'_{E1}) \wedge (Q_D \wedge Q'_{E2}) \\
\hline
(Q'_C \wedge Q_D) \wedge Q'_E = (Q'_C \wedge Q_D) \wedge Q'_E
\end{array}
\begin{array}{l}
\\ \text{EXPAND} \\ \text{EXPAND} \\ \wedge\ \text{DIST}
\end{array}
$$

   iii. *C, D, and E interfere. Under the actual set of assumptions, D and E communicate. According to case 2c, the property does not hold if D and E communicate*

(c) *C communicates with E. According to case 2c, the property does not hold if D and E communicate*

(d) *C communicates both with D and E*

   i. *C and D interfere. Under the actual set of assumptions, D and E communicate. According to case 2c, the property does not hold if D and E communicate*

   ii. *C interferes with E. Let Q_{C_i}, Q_{D_j}, Q_{E_k}, Q_{E1_h}, Q_{E1_l} be clauses of Q_C, Q_D, Q_E, Q_{E1}, and Q_{E2}, respectively. Assume that Q_{C_i} includes a reference to the post-state value of s, that Q_{D_j} includes a reference to both the pre-state and post-state values of s, and also that Q_{E_k}, Q_{E1_h}, and Q_{E1_l} include a reference to the pre-state values of s. According to the definition of $\blacktriangle$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/r'] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q'_{D_j}[r'/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q'_{E1} \triangleq Q_{E1_1} \wedge \ldots \wedge Q_{E1_h}[r/y] \wedge \ldots \wedge Q_{E1_o}$$
$$Q'_{E2} \triangleq Q_{E2_1} \wedge \ldots \wedge Q_{E2_l}[r/y] \wedge \ldots \wedge Q_{E2_r}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[r/y] \wedge \ldots \wedge Q_{E_p}$$

$$
\begin{array}{c}
(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E}_1)\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}_2) \\
\hline
(Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} = (Q'_C \wedge Q'_{E1}, \ldots)\blacktriangle(Q_D \wedge Q_{E2}, \ldots) \\
\hline
(Q'_C \wedge Q''_D) \wedge Q'_E = (Q'_C \wedge Q'_{E1}) \wedge (Q''_D \wedge Q'_{E2}) \\
\hline
(Q'_C \wedge Q''_D) \wedge Q'_E = (Q'_C \wedge Q''_D) \wedge Q'_E
\end{array}
\begin{array}{l}
\\ \text{EXPAND} \\ \text{EXPAND} \\ \wedge\ \text{DIST}
\end{array}
$$

   iii. *D and E interfere. Under the actual set of assumptions, D and E communicate. According to case 2c, the property does not hold if D and E communicate*

   iv. *C, D, and E interfere. Under the actual set of assumptions, D and E communicate. According to case 2c, the property does not hold if D and E communicate*

Lemma 9 (Left Distributivity of $\triangle$ over $\barwedge$). *Consider $\mathcal{C} \triangleq (P_C, R_C, W_C)$, $\mathcal{D} \triangleq (P_D, R_D, W_D)$, and $\mathcal{E} \triangleq (P_E, R_E, W_E)$ as the precondition partial contracts of Orc expressions C, D, and D, respectively. $\triangle$ is* ***left distributive*** *over $\barwedge$ i.e.,*

$$\mathcal{C}\triangle(\mathcal{D} \barwedge \mathcal{E}) = \mathcal{C}\triangle\mathcal{D} \barwedge \mathcal{C}\triangle\mathcal{E}$$

Proof. *Since $\barwedge$ does not denote order of execution, D and E are independent and do not interfere. Hence, consider the following two cases:*

1. *C is independent from both D and E*

$$\dfrac{\dfrac{\mathcal{C}\triangle(\mathcal{D}\barwedge\mathcal{E}) = \mathcal{C}\triangle\mathcal{D}\barwedge\mathcal{C}\triangle\mathcal{E}}{P_C \wedge (P_D \wedge P_E) = (P_C \wedge P_D) \wedge (P_C \wedge P_E)}\ \text{EXPAND}}{P_C \wedge (P_D \wedge P_E) = P_C \wedge (P_D \wedge P_E)}\ \wedge\ \text{DIST}$$

2. *Expressions communicate via a location s*
   (a) *C communicates either with D or E. Assume that C communicates with D[3]. Let P_{D_i} be the clause of P_D that includes a reference to s. According to the definitions of $\triangle$ and $\barwedge$,*

$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{D_n}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{C}\triangle(\mathcal{D}\barwedge\mathcal{E}) = \mathcal{C}\triangle\mathcal{D}\barwedge\mathcal{C}\triangle\mathcal{E}}{\mathcal{C}\triangle(P_D \wedge P_E, \ldots) = ((P_C \wedge P'_D) \barwedge (P_C \wedge P_E), \ldots)}\ \text{EXPAND}}{P_C \wedge (P'_D \wedge P_E) = (P_C \wedge P'_D) \wedge (P_C \wedge P_E)}\ \text{EXPAND}}{P_C \wedge (P'_D \wedge P_E) = P_C \wedge (P'_D \wedge P_E)}\ \wedge\ \text{DIST}$$

   (b) *C communicates with both D and E. Let P_{D_i} and P_{E_j} be the clauses of P_D and P_E, respectively, that includes a reference to s. According to the definitions of $\triangle$ and $\barwedge$,*

$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{D_n}$$
$$P'_E \triangleq P_{E_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E_m}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{C}\triangle(\mathcal{D}\barwedge\mathcal{E}) = \mathcal{C}\triangle\mathcal{D}\barwedge\mathcal{C}\triangle\mathcal{E}}{\mathcal{C}\triangle(P_D \wedge P_E, \ldots) = ((P_C \wedge P'_D) \barwedge (P_C \wedge P'_E), \ldots)}\ \text{EXPAND}}{P_C \wedge (P'_D \wedge P'_E) = (P_C \wedge P'_D) \wedge (P_C \wedge P'_E)}\ \text{EXPAND}}{P_C \wedge (P'_D \wedge P'_E) = P_C \wedge (P'_D \wedge P'_E)}\ \wedge\ \text{DIST}$$

Lemma 10 (Left Distributivity of $\blacktriangle$ over $\veebar$). *Consider $\mathcal{C} \triangleq (Q_C, R_C, W_C)$, $\mathcal{D} \triangleq (Q_D, R_D, W_D)$, and $\mathcal{E} \triangleq (Q_E, R_E, W_E)$ as the postcondition partial contracts of Orc expressions C, D, and D, respectively. $\blacktriangle$ is **left distributive** over $\veebar$, i.e.,*

$$\mathcal{C}\blacktriangle(\mathcal{D}\veebar\mathcal{E}) = \mathcal{C}\blacktriangle\mathcal{D}\veebar\mathcal{C}\blacktriangle\mathcal{E}$$

Proof. *Since $\veebar$ does not denote order of execution, D and E are independent and do not interfere. Hence, consider the following four cases:*

[3] Since $\barwedge$ is commutative, the two expressions under comparison are symmetric, and thus this case reduction is legitimate.

1. *Expressions D and E are independent from C and non-interfering*

$$\begin{array}{c}
\mathcal{C}\blacktriangle(\mathcal{D}\veebar\mathcal{E}) = \mathcal{C}\blacktriangle\mathcal{D}\veebar\mathcal{C}\blacktriangle\mathcal{E} \\ \hline
\mathcal{C}\blacktriangle(Q_D \vee Q_E, \ldots) = (Q_C \wedge Q_D, \ldots) \veebar (Q_C \wedge Q_E, \ldots) \\ \hline
Q_C \wedge (Q_D \vee Q_E) = (Q_C \wedge Q_D) \vee (Q_C \wedge Q_E) \\ \hline
Q_C \wedge (Q_D \vee Q_E) = Q_C \wedge (Q_D \vee Q_E)
\end{array}
\begin{array}{l} \textsc{Expand} \\ \textsc{Expand} \\ \textsc{Dist}\ \wedge/\vee \end{array}$$

2. *Expressions communicate via a location s and do not interfere*
   (a) *C communicates either with D or E. Assume that C communicates with D*[4]*. Let Q_{D_i} be the clause of Q_D that includes a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$

$$\begin{array}{c}
\mathcal{C}\blacktriangle(\mathcal{D}\veebar\mathcal{E}) = \mathcal{C}\blacktriangle\mathcal{D}\veebar\mathcal{C}\blacktriangle\mathcal{E} \\ \hline
\mathcal{C}\blacktriangle(Q_D \vee Q_E, \ldots) = (Q_C \wedge Q'_D, \ldots) \veebar (Q_C \wedge Q_E, \ldots) \\ \hline
Q_C \wedge (Q'_D \vee Q_E) = (Q_C \wedge Q'_D) \vee (Q_C \wedge Q_E) \\ \hline
Q_C \wedge (Q'_D \vee Q_E) = Q_C \wedge (Q'_D \vee Q_E)
\end{array}
\begin{array}{l} \textsc{Expand} \\ \textsc{Expand} \\ \textsc{Dist}\ \wedge/\vee \end{array}$$

   (b) *C communicates both with D and E. Let Q_{D_i} and Q_{E_j} be the clauses of Q_D and Q_E, respectively, that include a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_j}[r/r'] \wedge \ldots \wedge Q_{E_m}$$

$$\begin{array}{c}
\mathcal{C}\blacktriangle(\mathcal{D}\veebar\mathcal{E}) = \mathcal{C}\blacktriangle\mathcal{D}\veebar\mathcal{C}\blacktriangle\mathcal{E} \\ \hline
\mathcal{C}\blacktriangle(Q_D \vee Q_E, \ldots) = (Q_C \wedge Q'_D, \ldots) \veebar (Q_C \wedge Q'_E, \ldots) \\ \hline
Q_C \wedge (Q'_D \vee Q'_E) = (Q_C \wedge Q'_D) \vee (Q_C \wedge Q'_E) \\ \hline
Q_C \wedge (Q'_D \vee Q'_E) = Q_C \wedge (Q'_D \vee Q'_E)
\end{array}
\begin{array}{l} \textsc{Expand} \\ \textsc{Expand} \\ \textsc{Dist}\ \wedge/\vee \end{array}$$

3. *Expressions do not communicate and interfere via a location s*
   (a) *C interferes either with D or E. Assume that C interferes with D. Let Q_{C_i} be the clause of Q_C that includes a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$

$$\begin{array}{c}
\mathcal{C}\blacktriangle(\mathcal{D}\veebar\mathcal{E}) = \mathcal{C}\blacktriangle\mathcal{D}\veebar\mathcal{C}\blacktriangle\mathcal{E} \\ \hline
\mathcal{C}\blacktriangle(Q_D \vee Q_E, \ldots) = (Q'_C \wedge Q_D, \ldots) \veebar (Q_C \wedge Q_E, \ldots) \\ \hline
Q'_C \wedge (Q_D \vee Q_E) = (Q'_C \wedge Q_D) \vee (Q_C \wedge Q_E) \\ \hline
Q'_C \wedge (Q_D \vee Q_E) = Q'_C \wedge (Q_D \vee Q_E)
\end{array}
\begin{array}{l} \textsc{Expand} \\ \textsc{Expand} \\ \textsc{Dist}\ \wedge/\vee \end{array}$$

---

[4] Since $\veebar$ is commutative, the two expressions under comparison are symmetric, and thus this case reduction is legitimate. The same principle applies in the case C interferes either with D or E

(b) *C interferes both with D and E. Let Q_{C_i} be the clause of Q_C that includes a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{C_n}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{C}\blacktriangle(\mathcal{D} \veebar \mathcal{E}) = \mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{C}\blacktriangle\mathcal{E}}{\mathcal{C}\blacktriangle(Q_D \vee Q_E, \ldots) = (Q'_C \wedge Q_D, \ldots) \veebar (Q'_C \wedge Q_E, \ldots)}\ \text{EXPAND}}{Q'_C \wedge (Q_D \vee Q_E) = (Q'_C \wedge Q_D) \vee (Q'_C \wedge Q_E)}\ \text{EXPAND}}{Q'_C \wedge (Q_D \vee Q_E) = Q'_C \wedge (Q_D \vee Q_E)}\ \text{DIST } \wedge/\vee$$

4. *Expressions communicate and interfere via a location s. Since D and E cannot communicate or interfere, all the subcases where C communicates and interferes either with D and E are captured as part of the general case where C communicates and interferes both with D and E. Thus, assume the latter case. Let Q_{C_i}, Q_{D_j}, and Q_{E_k} be clauses of Q_C, Q_D, and Q_E, respectively. Assume that Q_{C_i} includes a reference to the post-state value of s, and that Q_{D_j} and Q_{E_k} include a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m}$$

$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[r/y] \wedge \ldots \wedge Q_{E_o}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{C}\blacktriangle(\mathcal{D} \veebar \mathcal{E}) = \mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{C}\blacktriangle\mathcal{E}}{\mathcal{C}\blacktriangle(Q_D \vee Q_E, \ldots) = (Q'_C \wedge Q'_D, \ldots) \veebar (Q'_C \wedge Q'_E, \ldots)}\ \text{EXPAND}}{Q'_C \wedge (Q'_D \vee Q'_E) = (Q'_C \wedge Q'_D) \vee (Q'_C \wedge Q'_E)}\ \text{EXPAND}}{Q'_C \wedge (Q'_D \vee Q'_E) = Q'_C \wedge (Q'_D \vee Q'_E)}\ \text{DIST } \wedge/\vee$$

Lemma 11 (Right Distributivity of $\triangle$ over $\overline{\wedge}$). *Let $\mathcal{C} \triangleq (P_C, R_C, W_C)$, $\mathcal{D} \triangleq (Q_D, R_D, W_D)$, and $\mathcal{E} \triangleq (Q_E, R_E, W_E)$ be the precondition partial contracts of Orc expressions C, D, and D, respectively. $\triangle$ is* ***right distributive*** *over $\overline{\wedge}$, i.e.,*

$$(\mathcal{C}\triangle\mathcal{D} \,\overline{\wedge}\, \mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} \,\overline{\wedge}\, (\mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E}$$

Proof. *Consider the following two cases:*

1. *Expressions C, D, and E are independent*

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\triangle\mathcal{D} \,\overline{\wedge}\, \mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} \,\overline{\wedge}\, (\mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E}}{\begin{array}{c}((P_C \wedge P_D, \ldots) \,\overline{\wedge}\, (P_D \wedge P_C, \ldots))\triangle\mathcal{E} = \\ (P_C \wedge P_D, \ldots)\triangle\mathcal{E} \,\overline{\wedge}\, (P_D \wedge P_C, \ldots)\triangle\mathcal{E}\end{array}}\ \text{EXPAND}}{\begin{array}{c}((P_C \wedge P_D) \wedge (P_D \wedge P_C), \ldots)\triangle\mathcal{E} = \\ (((P_C \wedge P_D) \wedge P_E), \ldots) \,\overline{\wedge}\, ((P_D \wedge P_C) \wedge P_E, \ldots)\end{array}}\ \text{EXPAND}}{\begin{array}{c}((P_C \wedge P_D) \wedge (P_D \wedge P_C)) \wedge P_E = \\ ((P_C \wedge P_D) \wedge P_E) \wedge ((P_D \wedge P_C) \wedge P_E)\end{array}}\ \text{EXPAND}}{((P_C \wedge P_D) \wedge (P_D \wedge P_C)) \wedge P_E = ((P_C \wedge P_D) \wedge (P_D \wedge P_C)) \wedge P_E}\ \wedge\ \text{DIST}$$

2. *Expressions communicate via a location s*

   (a) *Either C communicates with D, or D communicates with C. Assume that C communicates with D*[5]*. Let P_{D_i} be the clause of P_D that includes a reference to s. According to the definitions of $\triangle$ and $\overline{\wedge}$,*

$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{D_n}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\triangle\mathcal{D}\,\overline{\wedge}\,\mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E}\,\overline{\wedge}\,(\mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E}}{\begin{array}{c}((P_C \wedge P'_D, \ldots)\,\overline{\wedge}\,(P_D \wedge P_C, \ldots))\triangle\mathcal{E} = \\ (P_C \wedge P'_D, \ldots)\triangle\mathcal{E}\,\overline{\wedge}\,(P_D \wedge P_C, \ldots)\triangle\mathcal{E}\end{array}}\ \textsc{expand}}{\begin{array}{c}((P_C \wedge P'_D) \wedge (P_D \wedge P_C), \ldots)\triangle\mathcal{E} = \\ (((P_C \wedge P'_D) \wedge P_E), \ldots)\,\overline{\wedge}\,((P_D \wedge P_C) \wedge P_E, \ldots)\end{array}}\ \textsc{expand}}{\begin{array}{c}((P_C \wedge P'_D) \wedge (P_D \wedge P_C)) \wedge P_E = \\ ((P_C \wedge P'_D) \wedge P_E) \wedge ((P_D \wedge P_C) \wedge P_E)\end{array}}\ \textsc{expand}}{((P_C \wedge P'_D) \wedge (P_D \wedge P_C)) \wedge P_E = ((P_C \wedge P'_D) \wedge (P_D \wedge P_C)) \wedge P_E}\ \wedge\ \textsc{dist}$$

   (b) *C communicates with D, and D communicates with C. Let P_{C_i} and P_{D_j} be the clauses of P_C and P_D, respectively, that include a reference to s. According to the definitions of $\triangle$ and $\overline{\wedge}$,*

$$P'_C \triangleq P_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{C_n}$$
$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{D_m}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\triangle\mathcal{D}\,\overline{\wedge}\,\mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E}\,\overline{\wedge}\,(\mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E}}{\begin{array}{c}((P_C \wedge P'_D, \ldots)\,\overline{\wedge}\,(P_D \wedge P'_C, \ldots))\triangle\mathcal{E} = \\ (P_C \wedge P'_D, \ldots)\triangle\mathcal{E}\,\overline{\wedge}\,(P_D \wedge P'_C, \ldots)\triangle\mathcal{E}\end{array}}\ \textsc{expand}}{\begin{array}{c}((P_C \wedge P'_D) \wedge (P_D \wedge P'_C), \ldots)\triangle\mathcal{E} = \\ (((P_C \wedge P'_D) \wedge P_E), \ldots)\,\overline{\wedge}\,((P_D \wedge P'_C) \wedge P_E, \ldots)\end{array}}\ \textsc{expand}}{\begin{array}{c}((P_C \wedge P'_D) \wedge (P_D \wedge P'_C)) \wedge P_E = \\ ((P_C \wedge P'_D) \wedge P_E) \wedge ((P_D \wedge P'_C) \wedge P_E)\end{array}}\ \textsc{expand}}{((P_C \wedge P'_D) \wedge (P_D \wedge P'_C)) \wedge P_E = ((P_C \wedge P'_D) \wedge (P_D \wedge P'_C)) \wedge P_E}\ \wedge\ \textsc{dist}$$

   (c) *Either C or D communicates with E. Without loss of generality, according to lemma 1 and considering that the two expressions under comparison are symmetric, we just need to capture one of these subcases. Assume that C communicates with E. Let P_{E_i} be the clause of P_E that includes a reference to s. According to the definitions of $\triangle$ and $\overline{\wedge}$,*

$$P'_E \triangleq P_{E_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E_n}$$

---

[5] Since $\overline{\wedge}$ is commutative, the two expressions under comparison are symmetric, and thus this case reduction is legitimate.

$$
\begin{array}{cl}
(\mathcal{C}\triangle\mathcal{D} \barwedge \mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} \barwedge (\mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E} & \\
\hline
((P_C \wedge P_D, \ldots) \barwedge (P_D \wedge P_C, \ldots))\triangle\mathcal{E} = & \textsc{expand} \\
(P_C \wedge P_D, \ldots)\triangle\mathcal{E} \barwedge (P_D \wedge P_C, \ldots)\triangle\mathcal{E} & \\
\hline
((P_C \wedge P_D) \wedge (P_D \wedge P_C), \ldots)\triangle\mathcal{E} = & \textsc{expand} \\
(((P_C \wedge P_D) \wedge P'_E), \ldots) \barwedge ((P_D \wedge P_C) \wedge P'_E, \ldots) & \\
\hline
((P_C \wedge P_D) \wedge (P_D \wedge P_C)) \wedge P'_E = & \textsc{expand} \\
((P_C \wedge P_D) \wedge P'_E) \wedge ((P_D \wedge P_C) \wedge P'_E) & \\
\hline
((P_C \wedge P_D) \wedge (P_D \wedge P_C)) \wedge P'_E = ((P_C \wedge P_D) \wedge (P_D \wedge P_C)) \wedge P'_E & \wedge\ \textsc{dist}
\end{array}
$$

(d) *Either C communicates with both D and E, or D communicates with both C and E. Assume that C communicates with both D and E*[6]*. Let P_{D_i} and P_{E_j} be the clauses of P_D and P_E, respectively, that include a reference to s. According to the definitions of $\triangle$ and $\barwedge$,*

$$P'_D \triangleq P_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{D_n}$$

$$P'_E \triangleq P_{E_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge P_{E_m}$$

$$
\begin{array}{cl}
(\mathcal{C}\triangle\mathcal{D} \barwedge \mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} \barwedge (\mathcal{D}\triangle\mathcal{C})\triangle\mathcal{E} & \\
\hline
((P_C \wedge P'_D, \ldots) \barwedge (P_D \wedge P_C, \ldots))\triangle\mathcal{E} = & \textsc{expand} \\
(P_C \wedge P'_D, \ldots)\triangle\mathcal{E} \barwedge (P_D \wedge P_C, \ldots)\triangle\mathcal{E} & \\
\hline
((P_C \wedge P'_D) \wedge (P'_D \wedge P_C), \ldots)\triangle\mathcal{E} = & \textsc{expand} \\
(((P_C \wedge P'_D) \wedge P'_E), \ldots) \barwedge ((P_D \wedge P_C) \wedge P'_E, \ldots) & \\
\hline
((P_C \wedge P'_D) \wedge (P'_D \wedge P_C)) \wedge P'_E = & \textsc{expand} \\
((P_C \wedge P'_D) \wedge P'_E) \wedge ((P'_D \wedge P_C) \wedge P'_E) & \\
\hline
((P_C \wedge P'_D) \wedge (P'_D \wedge P_C)) \wedge P'_E = ((P_C \wedge P'_D) \wedge (P'_D \wedge P_C)) \wedge P'_E & \wedge\ \textsc{dist}
\end{array}
$$

Lemma 12 (Right Distributivity of $\blacktriangle$ over $\veebar$). *Assume $\mathcal{C} \triangleq (Q_C, R_C, W_C)$, $\mathcal{D} \triangleq (Q_D, R_D, W_D)$, and $\mathcal{E} \triangleq (Q_E, R_E, W_E)$ as the postcondition partial contracts of Orc expressions C, D, and D, respectively. $\blacktriangle$ is **right distributive** over $\veebar$, i.e.,*

$$(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}$$

Proof. *Consider the following four cases:*

1. *C, D, and E are independent and non-interfering*

$$
\begin{array}{cl}
(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} & \\
\hline
((Q_C \wedge Q_D, \ldots) \veebar (Q_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = & \textsc{expand} \\
(Q_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E} & \\
\hline
((Q_C \wedge Q_D) \vee (Q_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = & \textsc{expand} \\
(((Q_C \wedge Q_D) \wedge Q_E), \ldots) \veebar ((Q_D \wedge Q_C) \wedge Q_E, \ldots) & \\
\hline
((Q_C \wedge Q_D) \vee (Q_D \wedge Q_C)) \wedge Q_E = & \textsc{expand} \\
((Q_C \wedge Q_D) \wedge Q_E) \vee ((Q_D \wedge Q_C) \wedge Q_E) & \\
\hline
((Q_C \wedge Q_D) \vee (Q_D \wedge Q_C)) \wedge Q_E = ((Q_C \wedge Q_D) \vee (Q_D \wedge Q_C)) \wedge Q_E & \textsc{dist}\ \wedge/\vee
\end{array}
$$

[6] Since $\barwedge$ is commutative, the two expressions under comparison are symmetric, and thus this case reduction is legitimate.

2. *Expressions communicate via a location s and do not interfere*

   (a) *Either C communicates with D, or D communicates with C. Assume that C communicates with D*[7]*. Let Q_{D_i} be the clause of Q_D that includes a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q_C \wedge Q'_D, \ldots) \veebar (Q_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}}\ \text{EXPAND}}{\begin{array}{c}((Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q_C \wedge Q'_D) \wedge Q_E), \ldots) \veebar ((Q_D \wedge Q_C) \wedge Q_E, \ldots)\end{array}}\ \text{EXPAND}}{\begin{array}{c}((Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C)) \wedge Q_E = \\ ((Q_C \wedge Q'_D) \wedge Q_E) \vee ((Q_D \wedge Q_C) \wedge Q_E\end{array}}\ \text{EXPAND}}{((Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C)) \wedge Q_E = ((Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C)) \wedge Q_E}\ \text{DIST} \wedge/\vee$$

   (b) *Either C or D communicates with E. Without loss of generality, according to lemma 2 and considering that the two expressions under comparison are symmetric, we just need to capture one of these subcases. Assume that C communicates with E. Let Q_{E_i} be the clause of Q_E that includes a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_i}[r/r'] \wedge \ldots \wedge Q_{E_n}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q_C \wedge Q_D, \ldots) \veebar (Q_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}}\ \text{EXPAND}}{\begin{array}{c}((Q_C \wedge Q_D) \vee (Q_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q_C \wedge Q_D) \wedge Q'_E), \ldots) \veebar ((Q_D \wedge Q_C) \wedge Q'_E, \ldots)\end{array}}\ \text{EXPAND}}{\begin{array}{c}((Q_C \wedge Q_D) \vee (Q_D \wedge Q_C)) \wedge Q'_E = \\ ((Q_C \wedge Q_D) \wedge Q'_E) \vee ((Q_D \wedge Q_C) \wedge Q'_E)\end{array}}\ \text{EXPAND}}{((Q_C \wedge Q_D) \vee (Q_D \wedge Q_C)) \wedge Q'_E = ((Q_C \wedge Q_D) \vee (Q_D \wedge Q_C)) \wedge Q'_E}\ \text{DIST} \wedge/\vee$$

   (c) *Either C communicates with both D and E, or D communicates with both C and E. Assume that C communicates with both D and E*[8]*. Let Q_{D_i} and Q_{E_j} be the clauses of Q_D and Q_E, respectively, that include*

[7] Since $\veebar$ is commutative, the two expressions under comparison are symmetric, and thus this case reduction is legitimate.

[8] Since $\veebar$ is commutative, the two expressions under comparison are symmetric, and thus this case reduction is legitimate.

a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_j}[r/r'] \wedge \ldots \wedge Q_{E_m}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q_C \wedge Q'_D, \ldots) \veebar (Q_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}}\ \textsc{expand}}{\begin{array}{c}(Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C)\blacktriangle\mathcal{E} = \\ (((Q_C \wedge Q'_D) \wedge Q'_E), \ldots) \veebar ((Q_D \wedge Q_C) \wedge Q'_E, \ldots)\end{array}}\ \textsc{expand}}{\begin{array}{c}((Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C)) \wedge Q'_E = \\ ((Q_C \wedge Q'_D) \wedge Q'_E) \vee ((Q_D \wedge Q_C) \wedge Q'_E)\end{array}}\ \textsc{expand}}{((Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C)) \wedge Q'_E = ((Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C)) \wedge Q'_E}\ \textsc{dist}\ \wedge/\vee$$

3. *Expressions do not communicate and interfere via a location s*

   (a) *C and D interfere. Let Q_{C_i} and Q_{D_j} be the clauses of Q_C and Q_D, respectively, that include a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{D_m}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q'_C \wedge Q_D, \ldots) \veebar (Q'_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q'_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q'_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}}\ \textsc{expand}}{\begin{array}{c}((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q'_C \wedge Q_D) \wedge Q_E), \ldots) \veebar ((Q'_D \wedge Q_C) \wedge Q_E, \ldots)\end{array}}\ \textsc{expand}}{\begin{array}{c}((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C)) \wedge Q_E = \\ ((Q'_C \wedge Q_D) \wedge Q_E) \vee ((Q'_D \wedge Q_C) \wedge Q_E)\end{array}}\ \textsc{expand}}{((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C)) \wedge Q_E = ((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C)) \wedge Q_E}\ \textsc{dist}\ \wedge/\vee$$

   (b) *Either C or D interferes with E. Without loss of generality, according to lemma 2 and considering that the two expressions under comparison are symmetric, we just need to capture one of these subcases. Assume that C interferes with E. Let Q_{C_i} be the clause of Q_C that includes a reference to the post-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{((Q'_C \wedge Q_D, \ldots) \veebar (Q_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = (Q'_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}}\ \textsc{expand}}{((Q'_C \wedge Q_D) \vee (Q_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = (((Q'_C \wedge Q_D) \wedge Q_E), \ldots) \veebar ((Q_D \wedge Q'_C) \wedge Q_E, \ldots)}\ \textsc{expand}}{((Q'_C \wedge Q_D) \vee (Q_D \wedge Q'_C)) \wedge Q_E = ((Q'_C \wedge Q_D) \wedge Q_E) \vee ((Q_D \wedge Q'_C) \wedge Q_E)}\ \textsc{expand}}{((Q'_C \wedge Q_D) \vee (Q_D \wedge Q'_C)) \wedge Q_E = ((Q'_C \wedge Q_D) \vee (Q_D \wedge Q'_C)) \wedge Q_E}\ \textsc{dist}\ \wedge/\vee$$

(c) *C, D, and E interfere. Let Q_{C_i} and Q_{D_j} be the clauses of Q_C and Q_D, respectively, that include a reference to the post-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{D_m}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{((Q'_C \wedge Q_D, \ldots) \veebar (Q'_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = (Q'_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q'_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}}\ \textsc{expand}}{((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = (((Q'_C \wedge Q'_D) \wedge Q_E), \ldots) \veebar ((Q'_D \wedge Q'_C) \wedge Q_E, \ldots)}\ \textsc{expand}}{((Q'_C \wedge Q'_D) \vee (Q'_D \wedge Q'_C)) \wedge Q_E = ((Q'_C \wedge Q'_D) \wedge Q_E) \vee ((Q'_D \wedge Q'_C) \wedge Q_E)}\ \textsc{expand}}{((Q'_C \wedge Q'_D) \vee (Q'_D \wedge Q'_C)) \wedge Q_E = ((Q'_C \wedge Q'_D) \vee (Q'_D \wedge Q'_C)) \wedge Q_E}\ \textsc{dist}\ \wedge/\vee$$

4. *Expressions communicate and interfere via a location s*
   (a) *Either C communicates with D, or C communicates with D. Assume that C communicates with D*
      i. *C and D interfere. Let Q_{C_i} and Q_{D_j} be clauses of Q_C and Q_D, respectively. Assume that Q_{C_i} includes a reference to the post-state value of s, and that Q_{D_j} includes a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{D_m}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{((Q'_C \wedge Q'_D, \ldots) \veebar (Q''_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = (Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q''_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}}\ \textsc{expand}}{((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = (((Q'_C \wedge Q'_D) \wedge Q_E), \ldots) \veebar ((Q''_D \wedge Q_C) \wedge Q_E, \ldots)}\ \textsc{expand}}{((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q_C)) \wedge Q_E = ((Q'_C \wedge Q'_D) \wedge Q_E) \vee ((Q''_D \wedge Q_C) \wedge Q_E)}\ \textsc{expand}}{((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q_C)) \wedge Q_E = ((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q_C)) \wedge Q_E}\ \textsc{dist}\ \wedge/\vee$$

ii. *Either C or D interferes with E. Assume that C interferes with E. Let Q_{C_i} and Q_{D_j} be clauses of Q_C and Q_D, respectively. Assume that Q_{C_i} includes a reference to the post-state value of s, and that Q_{D_j} includes a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/r'] \wedge \ldots \wedge Q_{D_m}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q_C \wedge Q'_D, \ldots) \veebar (Q_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}}\;\textsc{expand}}{\begin{array}{c}((Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q'_C \wedge Q'_D) \wedge Q_E), \ldots) \veebar ((Q_D \wedge Q'_C) \wedge Q_E, \ldots)\end{array}}\;\textsc{expand}}{\begin{array}{c}((Q'_C \wedge Q'_D) \vee (Q_D \wedge Q'_C)) \wedge Q_E = \\ ((Q'_C \wedge Q'_D) \wedge Q_E) \vee ((Q_D \wedge Q'_C) \wedge Q_E)\end{array}}\;\textsc{expand}}{((Q'_C \wedge Q'_D) \vee (Q_D \wedge Q'_C)) \wedge Q_E = ((Q'_C \wedge Q'_D) \vee (Q_D \wedge Q'_C)) \wedge Q_E}\;\textsc{dist}\ \wedge/\vee$$

iii. *C, D, and E interfere. Let Q_{C_i} and Q_{D_j} be clauses of Q_C and Q_D, respectively. Assume that Q_{C_i} includes a reference to the post-state value of s, and that Q_{D_j} includes a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q''_C \triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{D_m}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q'_C \wedge Q'_D, \ldots) \veebar (Q''_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q''_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}}\;\textsc{expand}}{\begin{array}{c}((Q'_C \wedge Q'_D) \vee (Q'_D \wedge Q'_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q''_C \wedge Q''_D) \wedge Q_E), \ldots) \veebar ((Q''_D \wedge Q''_C) \wedge Q_E, \ldots)\end{array}}\;\textsc{expand}}{\begin{array}{c}((Q''_C \wedge Q''_D) \vee (Q''_D \wedge Q''_C)) \wedge Q_E = \\ ((Q''_C \wedge Q''_D) \wedge Q_E) \vee ((Q''_D \wedge Q''_C) \wedge Q_E)\end{array}}\;\textsc{expand}}{((Q''_C \wedge Q''_D) \vee (Q''_D \wedge Q''_C)) \wedge Q_E = ((Q''_C \wedge Q''_D) \vee (Q''_D \wedge Q''_C)) \wedge Q_E}\;\textsc{dist}\ \wedge/\vee$$

(b) *C communicates with D, and D communicates with C. Under the actual set of assumptions, C and D interfere.*

i. *Let Q_{C_i} and Q_{D_j} be clauses of Q_C and Q_D, respectively. Assume that both Q_{C_i} and Q_{D_j} include references to the pre-state and post-state* *values of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$
\begin{aligned}
Q'_C &\triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n} \\
Q''_C &\triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r/y] \wedge \ldots \wedge Q_{C_n} \\
Q'_D &\triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m} \\
Q''_D &\triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r'/y] \wedge \ldots \wedge Q_{D_m}
\end{aligned}
$$

$$\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{gathered}((Q'_C \wedge Q'_D, \ldots) \veebar (Q''_D \wedge Q''_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q''_D \wedge Q''_C, \ldots)\blacktriangle\mathcal{E}\end{gathered}}\ \text{EXPAND}$$

$$\dfrac{\begin{gathered}((Q'_C \wedge Q'_D, \ldots) \veebar (Q''_D \wedge Q''_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q''_D \wedge Q''_C, \ldots)\blacktriangle\mathcal{E}\end{gathered}}{\begin{gathered}((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q''_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q'_C \wedge Q'_D) \wedge Q_E), \ldots) \veebar ((Q''_D \wedge Q''_C) \wedge Q_E, \ldots)\end{gathered}}\ \text{EXPAND}$$

$$\dfrac{\begin{gathered}((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q''_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q'_C \wedge Q'_D) \wedge Q_E), \ldots) \veebar ((Q''_D \wedge Q''_C) \wedge Q_E, \ldots)\end{gathered}}{\begin{gathered}((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q''_C)) \wedge Q_E = \\ ((Q'_C \wedge Q'_D) \wedge Q_E) \vee ((Q''_D \wedge Q''_C) \wedge Q_E)\end{gathered}}\ \text{EXPAND}$$

$$\dfrac{\begin{gathered}((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q''_C)) \wedge Q_E = \\ ((Q'_C \wedge Q'_D) \wedge Q_E) \vee ((Q''_D \wedge Q''_C) \wedge Q_E)\end{gathered}}{((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q''_C)) \wedge Q_E = ((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q''_C)) \wedge Q_E}\ \text{DIST } \wedge/\vee$$

ii. *Either C or D interferes with E. Hence, under the actual set of assumptions, C, D and E interfere. Assume that C interferes with E. Let Q_{C_i} and Q_{D_j} be clauses of Q_C and Q_D, respectively. Assume that both Q_{C_i} and Q_{D_j} include references to the pre-state and post-state values of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$
\begin{aligned}
Q'_C &\triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n} \\
Q''_C &\triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r/y] \wedge \ldots \wedge Q_{C_n} \\
Q'''_C &\triangleq Q_{C_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{C_n} \\
Q'_D &\triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y] \wedge \ldots \wedge Q_{D_m} \\
Q''_D &\triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r'/y] \wedge \ldots \wedge Q_{D_m} \\
Q'''_D &\triangleq Q_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{D_m}
\end{aligned}
$$

$$\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{gathered}((Q'_C \wedge Q'_D, \ldots) \veebar (Q''_D \wedge Q''_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q''_D \wedge Q''_C, \ldots)\blacktriangle\mathcal{E}\end{gathered}}\ \text{EXPAND}$$

$$\dfrac{\begin{gathered}((Q'_C \wedge Q'_D, \ldots) \veebar (Q''_D \wedge Q''_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q''_D \wedge Q''_C, \ldots)\blacktriangle\mathcal{E}\end{gathered}}{\begin{gathered}((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q''_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q'''_C \wedge Q'''_D) \wedge Q_E), \ldots) \veebar ((Q'''_D \wedge Q'''_C) \wedge Q_E, \ldots)\end{gathered}}\ \text{EXPAND}$$

$$\dfrac{\begin{gathered}((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q''_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q'''_C \wedge Q'''_D) \wedge Q_E), \ldots) \veebar ((Q'''_D \wedge Q'''_C) \wedge Q_E, \ldots)\end{gathered}}{\begin{gathered}((Q'''_C \wedge Q'''_D) \vee (Q'''_D \wedge Q'''_C)) \wedge Q_E = \\ ((Q'''_C \wedge Q'''_D) \wedge Q_E) \vee ((Q'''_D \wedge Q'''_C) \wedge Q_E)\end{gathered}}\ \text{EXPAND}$$

$$\dfrac{\begin{gathered}((Q'''_C \wedge Q'''_D) \vee (Q'''_D \wedge Q'''_C)) \wedge Q_E = \\ ((Q'''_C \wedge Q'''_D) \wedge Q_E) \vee ((Q'''_D \wedge Q'''_C) \wedge Q_E)\end{gathered}}{((Q'''_C \wedge Q'''_D) \vee (Q'''_D \wedge Q'''_C)) \wedge Q_E = ((Q'''_C \wedge Q'''_D) \vee (Q'''_D \wedge Q'''_C)) \wedge Q_E}\ \text{DIST } \wedge/\vee$$

(c) *Either C or D communicates with E. Assume that C communicates with E*

i. *C and D interfere. Under the actual set of assumptions, D communicates with E. Let Q_{C_i}, Q_{D_j}, and Q_{E_k} be clauses of Q_C, Q_D,* *and Q_E, respectively. Assume that Q_{C_i} and Q_{D_j} include references to the post-state value of s, and that Q_{E_k} includes references to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{D_m}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[r/r'] \wedge \ldots \wedge Q_{E_o}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q'_C \wedge Q_D,\ldots) \veebar (Q'_D \wedge Q_C,\ldots))\blacktriangle\mathcal{E} = \\ (Q'_C \wedge Q_D,\ldots)\blacktriangle\mathcal{E} \veebar (Q'_D \wedge Q_C,\ldots)\blacktriangle\mathcal{E}\end{array}}\ \text{EXPAND}}{\begin{array}{c}((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C),\ldots)\blacktriangle\mathcal{E} = \\ (((Q'_C \wedge Q_D) \wedge Q'_E),\ldots) \veebar ((Q'_D \wedge Q_C) \wedge Q'_E,\ldots)\end{array}}\ \text{EXPAND}}{\begin{array}{c}((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C)) \wedge Q'_E = \\ ((Q'_C \wedge Q_D) \wedge Q'_E) \vee ((Q'_D \wedge Q_C) \wedge Q'_E)\end{array}}\ \text{EXPAND}}{((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C)) \wedge Q'_E = ((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C)) \wedge Q'_E}\ \text{DIST}\ \wedge/\vee$$

ii. *Either C or D interferes with E. Assume that C interferes with E. Let Q_{C_i} and Q_{E_j} be clauses of Q_C and Q_E, respectively. Assume that Q_{C_i} includes a reference to the post-state value of s, and that Q_{E_j} includes a reference to the pre-state value of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_j}[r/y] \wedge \ldots \wedge Q_{E_m}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q_C \wedge Q_D,\ldots) \veebar (Q_D \wedge Q_C,\ldots))\blacktriangle\mathcal{E} = \\ (Q_C \wedge Q_D,\ldots)\blacktriangle\mathcal{E} \veebar (Q_D \wedge Q_C,\ldots)\blacktriangle\mathcal{E}\end{array}}\ \text{EXPAND}}{\begin{array}{c}((Q_C \wedge Q_D) \vee (Q_D \wedge Q_C),\ldots)\blacktriangle\mathcal{E} = \\ (((Q'_C \wedge Q_D) \wedge Q'_E),\ldots) \veebar ((Q_D \wedge Q'_C) \wedge Q'_E,\ldots)\end{array}}\ \text{EXPAND}}{\begin{array}{c}((Q'_C \wedge Q_D) \vee (Q_D \wedge Q'_C)) \wedge Q'_E = \\ ((Q'_C \wedge Q_D) \wedge Q_E) \vee ((Q_D \wedge Q'_C) \wedge Q'_E)\end{array}}\ \text{EXPAND}}{((Q'_C \wedge Q_D) \vee (Q_D \wedge Q'_C)) \wedge Q'_E = ((Q'_C \wedge Q_D) \vee (Q_D \wedge Q'_C)) \wedge Q'_E}\ \text{DIST}\ \wedge/\vee$$

iii. *C, D, and E interfere. Under the actual set of assumptions, D communicates with E. Let Q_{C_i}, Q_{D_j}, and Q_{E_k} be clauses of Q_C, Q_D, and Q_E, respectively. Assume that Q_{C_i} and Q_{D_j} include a reference to the post-state value of s, and that Q_{E_k} includes a reference to the* *pre-state value of s. According to the definitions of* $\blacktriangle$ *and* $\veebar$,

$$Q'_C \triangleq Q_{C_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{C_n}$$
$$Q''_C \triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n}$$
$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{D_m}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r'/y] \wedge \ldots \wedge Q_{D_m}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[r/y] \wedge \ldots \wedge Q_{E_o}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q'_C \wedge Q_D, \ldots) \veebar (Q'_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q'_C \wedge Q_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q'_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}} \text{EXPAND}}{\begin{array}{c}((Q'_C \wedge Q_D) \vee (Q'_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q'_C \wedge Q''_D) \wedge Q'_E), \ldots) \veebar ((Q'_D \wedge Q''_C) \wedge Q'_E, \ldots)\end{array}} \text{EXPAND}}{\begin{array}{c}((Q'_C \wedge Q''_D) \vee (Q'_D \wedge Q''_C)) \wedge Q'_E = \\ ((Q'_C \wedge Q''_D) \wedge Q'_E) \vee ((Q'_D \wedge Q''_C) \wedge Q'_E)\end{array}} \text{EXPAND}}{((Q'_C \wedge Q''_D) \vee (Q'_D \wedge Q''_C)) \wedge Q'_E = ((Q'_C \wedge Q''_D) \vee (Q'_D \wedge Q''_C)) \wedge Q'_E} \text{DIST } \wedge/\vee$$

(d) *Either C communicates with both D and E, or D communicates with both C and E. Assume that C communicates with both D and E*

i. *C and D interfere. Under the actual set of assumptions, D communicates with E. Let* Q_{D_i} *and* Q_{E_j} *be clauses of* Q_D *and* Q_E*, respectively. Assume that* Q_{D_i} *includes a reference to both the pre-state and post-state values of s, and that* Q_{E_j} *includes a reference to the pre-state value of s. According to the definitions of* $\blacktriangle$ *and* $\veebar$,

$$Q'_D \triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_i}[r/r'] \wedge \ldots \wedge Q_{D_n}$$
$$Q''_D \triangleq Q_{D_1} \wedge \ldots \wedge true \wedge \ldots \wedge Q_{D_n}$$
$$Q'_E \triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_j}[r/r'] \wedge \ldots \wedge Q_{E_m}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q_C \wedge Q'_D, \ldots) \veebar (Q''_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q''_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}} \text{EXPAND}}{\begin{array}{c}((Q_C \wedge Q'_D) \vee (Q''_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q_C \wedge Q'_D) \wedge Q'_E), \ldots) \veebar ((Q''_D \wedge Q_C) \wedge Q'_E, \ldots)\end{array}} \text{EXPAND}}{\begin{array}{c}((Q_C \wedge Q'_D) \vee (Q''_D \wedge Q_C)) \wedge Q'_E = \\ ((Q_C \wedge Q'_D) \wedge Q'_E) \vee ((Q''_D \wedge Q_C) \wedge Q'_E)\end{array}} \text{EXPAND}}{((Q_C \wedge Q'_D) \vee (Q''_D \wedge Q_C)) \wedge Q'_E = ((Q_C \wedge Q'_D) \vee (Q''_D \wedge Q_C)) \wedge Q'_E} \text{DIST } \wedge/\vee$$

ii. *Either C or D interferes with E. Assume that C interferes with E. Let* Q_{C_i}, Q_{D_j}*, and* Q_{E_k} *be the clauses of* Q_C, Q_D *and* Q_E*, respectively, that include a reference to both the pre-state and post-state* *values of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$
\begin{aligned}
Q'_C &\triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n} \\
Q'_D &\triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/r'] \wedge \ldots \wedge Q_{D_m} \\
Q''_D &\triangleq Q_{D_1} \wedge \ldots \wedge Q'_{D_j}[r'/y] \wedge \ldots \wedge Q_{D_m} \\
Q'_E &\triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[r/y] \wedge \ldots \wedge Q_{E_o}
\end{aligned}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q_C \wedge Q'_D, \ldots) \veebar (Q_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}}\ \textsc{expand}}{\begin{array}{c}((Q_C \wedge Q'_D) \vee (Q_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q'_C \wedge Q''_D) \wedge Q'_E), \ldots) \veebar ((Q_D \wedge Q'_C) \wedge Q'_E, \ldots)\end{array}}\ \textsc{expand}}{\begin{array}{c}((Q'_C \wedge Q''_D) \vee (Q_D \wedge Q'_C)) \wedge Q'_E = \\ ((Q'_C \wedge Q''_D) \wedge Q'_E) \vee ((Q_D \wedge Q'_C) \wedge Q'_E)\end{array}}\ \textsc{expand}}{((Q'_C \wedge Q''_D) \vee (Q_D \wedge Q'_C)) \wedge Q'_E = ((Q'_C \wedge Q''_D) \vee (Q_D \wedge Q'_C)) \wedge Q'_E}\ \textsc{dist}\ \wedge/\vee
$$

iii. *C, D, and E interfere. Under the actual set of assumptions, D communicates with E. Let Q_{C_i}, Q_{D_j}, and Q_{E_k} be the clauses of Q_C, Q_D and Q_E, respectively, that include a reference to both the pre-state and post-state values of s. According to the definitions of $\blacktriangle$ and $\veebar$,*

$$
\begin{aligned}
Q'_C &\triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/y] \wedge \ldots \wedge Q_{C_n} \\
Q''_C &\triangleq Q_{C_1} \wedge \ldots \wedge Q_{C_i}[r'/z] \wedge \ldots \wedge Q_{C_n} \\
Q'_D &\triangleq Q_{D_1} \wedge \ldots \wedge Q_{D_j}[r/y'] \wedge \ldots \wedge Q_{D_m} \\
Q''_D &\triangleq Q_{D_1} \wedge \ldots \wedge \mathit{true} \wedge \ldots \wedge Q_{D_m} \\
Q'''_D &\triangleq Q_{D_1} \wedge \ldots \wedge Q'_{D_j}[r'/z] \wedge \ldots \wedge Q_{D_m} \\
Q'_E &\triangleq Q_{E_1} \wedge \ldots \wedge Q_{E_k}[r/z] \wedge \ldots \wedge Q_{E_o}
\end{aligned}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D} \veebar \mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} \veebar (\mathcal{D}\blacktriangle\mathcal{C})\blacktriangle\mathcal{E}}{\begin{array}{c}((Q'_C \wedge Q'_D, \ldots) \veebar (Q''_D \wedge Q_C, \ldots))\blacktriangle\mathcal{E} = \\ (Q'_C \wedge Q'_D, \ldots)\blacktriangle\mathcal{E} \veebar (Q''_D \wedge Q_C, \ldots)\blacktriangle\mathcal{E}\end{array}}\ \textsc{expand}}{\begin{array}{c}((Q'_C \wedge Q'_D) \vee (Q''_D \wedge Q_C), \ldots)\blacktriangle\mathcal{E} = \\ (((Q'_C \wedge Q'''_D) \wedge Q'_E), \ldots) \veebar ((Q''_D \wedge Q''_C) \wedge Q'_E, \ldots)\end{array}}\ \textsc{expand}}{\begin{array}{c}((Q'_C \wedge Q'''_D) \vee (Q''_D \wedge Q''_C)) \wedge Q'_E = \\ ((Q'_C \wedge Q'''_D) \wedge Q'_E) \vee ((Q''_D \wedge Q''_C) \wedge Q'_E)\end{array}}\ \textsc{expand}}{((Q'_C \wedge Q'''_D) \vee (Q''_D \wedge Q''_C)) \wedge Q'_E = ((Q'_C \wedge Q'''_D) \vee (Q''_D \wedge Q''_C)) \wedge Q'_E}\ \textsc{dist}\ \wedge/\vee
$$

Lemma 13 (Precondition Partial Contract Swap). *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be the precondition partial contracts of Orc expressions C, D, and D, respectively. $\triangle$ allows to* ***swap*** *$\mathcal{D}$ and $\mathcal{E}$ when $\mathcal{C}$ is the first or the last precondition partial* *contract in compositions like:*

$$(\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{E})\triangle\mathcal{D} = \mathcal{C}\triangle(\mathcal{D}\triangle\mathcal{E}) = \mathcal{C}\triangle(\mathcal{E}\triangle\mathcal{D})$$
$$(\mathcal{D}\triangle\mathcal{E})\triangle\mathcal{C} = (\mathcal{E}\triangle\mathcal{D})\triangle\mathcal{C} = \mathcal{D}\triangle(\mathcal{E}\triangle\mathcal{C}) = \mathcal{E}\triangle(\mathcal{D}\triangle\mathcal{C})$$

if D and E are independent.

Proof. *We analyze the two cases separately*

1. *$\mathcal{C}$ is the first precondition partial contract in the composition*

$$\dfrac{\dfrac{(\mathcal{C}\triangle\mathcal{D})\triangle\mathcal{E} = (\mathcal{C}\triangle\mathcal{E})\triangle\mathcal{D} = \mathcal{C}\triangle(\mathcal{D}\triangle\mathcal{E}) = \mathcal{C}\triangle(\mathcal{E}\triangle\mathcal{D})}{\mathcal{C}\triangle(\mathcal{D}\triangle\mathcal{E}) = \mathcal{C}\triangle(\mathcal{E}\triangle\mathcal{D}) = \mathcal{C}\triangle(\mathcal{D}\triangle\mathcal{E}) = \mathcal{C}\triangle(\mathcal{E}\triangle\mathcal{D})}\ \triangle\ \textsc{assoc}}{\mathcal{C}\triangle(\mathcal{D}\triangle\mathcal{E}) = \mathcal{C}\triangle(\mathcal{D}\triangle\mathcal{E}) = \mathcal{C}\triangle(\mathcal{D}\triangle\mathcal{E}) = \mathcal{C}\triangle(\mathcal{D}\triangle\mathcal{E})}\ \triangle\ \textsc{comm}$$

2. *$\mathcal{C}$ is the last precondition partial contract in the composition*

$$\dfrac{\dfrac{(\mathcal{D}\triangle\mathcal{E})\triangle\mathcal{C} = (\mathcal{E}\triangle\mathcal{D})\triangle\mathcal{C} = \mathcal{D}\triangle(\mathcal{E}\triangle\mathcal{C}) = \mathcal{E}\triangle(\mathcal{D}\triangle\mathcal{C})}{(\mathcal{D}\triangle\mathcal{E})\triangle\mathcal{C} = (\mathcal{E}\triangle\mathcal{D})\triangle\mathcal{C} = (\mathcal{D}\triangle\mathcal{E})\triangle\mathcal{C} = (\mathcal{E}\triangle\mathcal{D})\triangle\mathcal{C}}\ \triangle\ \textsc{assoc}}{(\mathcal{D}\triangle\mathcal{E})\triangle\mathcal{C} = (\mathcal{D}\triangle\mathcal{E})\triangle\mathcal{C} = (\mathcal{D}\triangle\mathcal{E})\triangle\mathcal{C} = (\mathcal{D}\triangle\mathcal{E})\triangle\mathcal{C}}\ \triangle\ \textsc{comm}$$

Lemma 14 (Postcondition Partial Contract Swap). *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be the postcondition partial contracts of Orc expressions C, D, and D, respectively. $\blacktriangle$ allows to **swap** $\mathcal{D}$ and $\mathcal{E}$ when $\mathcal{C}$ is the first or the last postcondition partial contract in compositions like:*

$$(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E})\blacktriangle\mathcal{D} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}) = \mathcal{C}\blacktriangle(\mathcal{E}\blacktriangle\mathcal{D})$$
$$(\mathcal{D}\blacktriangle\mathcal{E})\blacktriangle\mathcal{C} = (\mathcal{E}\blacktriangle\mathcal{D})\blacktriangle\mathcal{C} = \mathcal{D}\blacktriangle(\mathcal{E}\blacktriangle\mathcal{C}) = \mathcal{E}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{C})$$

if D and E are independent and non-interfering.

Proof. *We analyze the two cases separately*

1. *$\mathcal{C}$ is the first postcondition partial contract in the composition*

$$\dfrac{\dfrac{(\mathcal{C}\blacktriangle\mathcal{D})\blacktriangle\mathcal{E} = (\mathcal{C}\blacktriangle\mathcal{E})\blacktriangle\mathcal{D} = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}) = \mathcal{C}\blacktriangle(\mathcal{E}\blacktriangle\mathcal{D})}{\mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}) = \mathcal{C}\blacktriangle(\mathcal{E}\blacktriangle\mathcal{D}) = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}) = \mathcal{C}\blacktriangle(\mathcal{E}\blacktriangle\mathcal{D})}\ \blacktriangle\ \textsc{assoc}}{\mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}) = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}) = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E}) = \mathcal{C}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{E})}\ \blacktriangle\ \textsc{comm}$$

2. *$\mathcal{C}$ is the last postcondition partial contract in the composition*

$$\dfrac{\dfrac{(\mathcal{D}\blacktriangle\mathcal{E})\blacktriangle\mathcal{C} = (\mathcal{E}\blacktriangle\mathcal{D})\blacktriangle\mathcal{C} = \mathcal{D}\blacktriangle(\mathcal{E}\blacktriangle\mathcal{C}) = \mathcal{E}\blacktriangle(\mathcal{D}\blacktriangle\mathcal{C})}{(\mathcal{D}\blacktriangle\mathcal{E})\blacktriangle\mathcal{C} = (\mathcal{E}\blacktriangle\mathcal{D})\blacktriangle\mathcal{C} = (\mathcal{D}\blacktriangle\mathcal{E})\blacktriangle\mathcal{C} = (\mathcal{E}\blacktriangle\mathcal{D})\blacktriangle\mathcal{C}}\ \blacktriangle\ \textsc{assoc}}{(\mathcal{D}\blacktriangle\mathcal{E})\blacktriangle\mathcal{C} = (\mathcal{D}\blacktriangle\mathcal{E})\blacktriangle\mathcal{C} = (\mathcal{D}\blacktriangle\mathcal{E})\blacktriangle\mathcal{C} = (\mathcal{D}\blacktriangle\mathcal{E})\blacktriangle\mathcal{C}}\ \blacktriangle\ \textsc{comm}$$

Besides, the reader may deduce from the definitions exposed previously that there is an equality relation between $\triangle$ and $\blacktriangle$, and the boolean *and* operator. Lemma 15 reveals such relation.

Lemma 15 (Equality between Boolean *and* and $\triangle$ and $\blacktriangle$). *Assume $\mathcal{C}_P \triangleq (P_C, R_C, W_C)$ and $\mathcal{D}_P \triangleq (P_D, R_D, W_D)$ as the precondition partial contracts of Orc expressions C, D, and D, respectively. In the case that expressions C and D are* ***independent****, then*

$$\mathcal{C}_P \triangle \mathcal{D}_P = P_C \wedge P_D$$

Let $\mathcal{C}_Q \triangleq (Q_C, R_C, W_C)$ and $\mathcal{D}_Q \triangleq (Q_D, R_D, W_D)$ be the postcondition partial contracts of Orc expressions C, D, and D, respectively. In the case that expressions C and D are ***independent*** *and* ***non-interfering****, then*

$$\mathcal{C}_Q \blacktriangle \mathcal{D}_Q = Q_C \wedge Q_D$$

B Correctness of Contract Derivation

This appendix provides the reader with a list of theorems that capture the algebraic properties that the semantic derivation function must satisfy to be considered correct. This list of theorems is showed in Section 4.2. Consider the following example.

Example 1 (Commutativity of Parallel Composition). Since the Orc parallel operator is commutative i.e.,

$$D \mid E = E \mid D$$

the semantic derivation Orc expressions must yield the same result:

$$[\![D \mid E]\!] = [\![E \mid D]\!]$$

To prove the algebraic properties of the semantic derivation function, we first expand the definitions of the Orc expressions under comparison. Figure 1 shows the result of expanding the definition of the semantic derivation of the expression $D \mid E$. The reader may realize the difficulty to reason about the expanded version of any composite expression. Thus, to reduce the syntactic complexity of the expressions under comparison, we opt for reusing the notation of the lemmas presented in Appendix A. Thus, the only symbols used by the new notation are partial contracts and the partial contract composition operators.

The following example shows an alternative version of the composite expression showed in Figure 1 written with the new notation.

Example 2 (Alternative Notation for Semantic Derivation). Let $C \triangleq D \mid E$. The behavioral semantic derivation of expression C results in a contract that includes the precondition and postcondition of C. We annotate the derived precondition partial contract of C as $\mathcal{C}_P$ and $[\![D \mid E]\!]_P$, and the derived postcondition partial contract of C as $\mathcal{C}_Q$ and $[\![D \mid E]\!]_Q$. Moreover, let $\mathcal{D}_P$ and $\mathcal{E}_P$ be the precondition partial contracts of expressions D and E, respectively. Likewise, let $\mathcal{D}_Q$ and $\mathcal{E}_Q$ be the postcondition partial contracts of expressions D and E, respectively. Thus,

$$\textsc{Precondition Derivation}\quad \dfrac{\dfrac{\mathcal{C}_P = [\![D \mid E]\!]_P}{(pre([\![D \gg E]\!]),\ldots) \mathbin{\bar{\wedge}} (pre([\![E \gg D]\!]),\ldots)}\ \textsc{expand}}{(\mathcal{D}_P \triangle \mathcal{E}_P) \mathbin{\bar{\wedge}} (\mathcal{E}_P \triangle \mathcal{D}_P)}\ \textsc{rewrite}$$

$$\textsc{Postcondition Derivation}\quad \dfrac{\dfrac{\mathcal{C}_Q = [\![D \mid E]\!]_Q}{(post([\![D \gg E]\!]),\ldots) \mathbin{\underline{\vee}} (post([\![E \gg D]\!]),\ldots)}\ \textsc{expand}}{(\mathcal{D}'_Q \blacktriangle \mathcal{E}_Q) \mathbin{\underline{\vee}} (\mathcal{E}'_Q \blacktriangle \mathcal{D}_Q)}\ \textsc{rewrite}$$

Note that the *prime* symbol over $\mathcal{D}'_Q$ and $\mathcal{E}'_Q$ denote that, according to Algorithm 3, the postcondition of expressions D and E, respectively, has been manipulated.

Fig. 1. Expansion of the Semantic Derivation of $D \mid E$.

$$
\begin{array}{c}
\llbracket D \mid E \rrbracket \\
\hline
(((\mathit{pre}(\llbracket D \gg E \rrbracket), \mathit{frame}(\llbracket D \rrbracket) \cup \mathit{frame}(\llbracket E \rrbracket), \\
\mathit{footprint}(\llbracket D \rrbracket) \cup \mathit{footprint}(\llbracket E \rrbracket))\overline{\wedge} \\
(\mathit{pre}(\llbracket E \gg D \rrbracket), \mathit{frame}(\llbracket E \rrbracket) \cup \mathit{frame}(\llbracket D \rrbracket), \\
\mathit{footprint}(\llbracket E \rrbracket) \cup \mathit{footprint}(\llbracket D \rrbracket)))'\mathit{assertion}, \\
((\mathit{post}(\llbracket D \gg E \rrbracket), \mathit{frame}(\llbracket D \rrbracket) \cup \mathit{frame}(\llbracket E \rrbracket), \\
\mathit{footprint}(\llbracket D \rrbracket) \cup \mathit{footprint}(\llbracket E \rrbracket))\underline{\vee} \\
(\mathit{post}(\llbracket E \gg D \rrbracket), \mathit{frame}(\llbracket E \rrbracket) \cup \mathit{frame}(\llbracket D \rrbracket), \\
\mathit{footprint}(\llbracket E \rrbracket) \cup \mathit{footprint}(\llbracket D \rrbracket)))'\mathit{assertion}, \\
\mathit{frame}(\llbracket D \rrbracket) \cup \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket D \rrbracket) \cup \mathit{footprint}(\llbracket E \rrbracket))
\end{array}\ \textsc{expand}
$$

$$
\begin{array}{c}
\hline
(((\mathit{pre}(((\mathit{pre}(\llbracket D \rrbracket), \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket D \rrbracket))\triangle \\
(\mathit{pre}(\llbracket E \rrbracket), \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket E \rrbracket)))'\mathit{assertion}, \\
((\mathit{exclRes}(\mathit{post}(\llbracket D \rrbracket)), \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket D \rrbracket))\blacktriangle \\
(\mathit{post}(\llbracket E \rrbracket), \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket E \rrbracket)))'\mathit{assertion}, \\
\mathit{frame}(\llbracket D \rrbracket) \cup \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket D \rrbracket) \cup \mathit{footprint}(\llbracket E \rrbracket)), \\
\mathit{frame}(\llbracket D \rrbracket) \cup \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket D \rrbracket) \cup \mathit{footprint}(\llbracket E \rrbracket))\overline{\wedge} \\
(\mathit{pre}(((\mathit{pre}(\llbracket E \rrbracket), \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket E \rrbracket))\triangle \\
(\mathit{pre}(\llbracket D \rrbracket), \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket D \rrbracket)))'\mathit{assertion}, \\
((\mathit{exclRes}(\mathit{post}(\llbracket E \rrbracket)), \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket E \rrbracket))\blacktriangle \\
(\mathit{post}(\llbracket D \rrbracket), \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket D \rrbracket)))'\mathit{assertion}, \\
\mathit{frame}(\llbracket E \rrbracket) \cup \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket E \rrbracket) \cup \mathit{footprint}(\llbracket D \rrbracket)), \\
\mathit{frame}(\llbracket E \rrbracket) \cup \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket E \rrbracket) \cup \mathit{footprint}(\llbracket D \rrbracket)))'\mathit{assertion}, \\
((\mathit{post}(((\mathit{pre}(\llbracket D \rrbracket), \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket D \rrbracket))\triangle \\
(\mathit{pre}(\llbracket E \rrbracket), \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket E \rrbracket)))'\mathit{assertion}, \\
((\mathit{exclRes}(\mathit{post}(\llbracket D \rrbracket)), \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket D \rrbracket))\blacktriangle \\
(\mathit{post}(\llbracket E \rrbracket), \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket E \rrbracket)))'\mathit{assertion}, \\
\mathit{frame}(\llbracket D \rrbracket) \cup \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket D \rrbracket) \cup \mathit{footprint}(\llbracket E \rrbracket)), \\
\mathit{frame}(\llbracket D \rrbracket) \cup \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket D \rrbracket) \cup \mathit{footprint}(\llbracket E \rrbracket))\underline{\vee} \\
(\mathit{post}(((\mathit{pre}(\llbracket E \rrbracket), \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket E \rrbracket))\triangle \\
(\mathit{pre}(\llbracket D \rrbracket), \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket D \rrbracket)))'\mathit{assertion}, \\
((\mathit{exclRes}(\mathit{post}(\llbracket E \rrbracket)), \mathit{frame}(\llbracket E \rrbracket), \mathit{footprint}(\llbracket E \rrbracket))\blacktriangle \\
(\mathit{post}(\llbracket D \rrbracket), \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket D \rrbracket)))'\mathit{assertion}, \\
\mathit{frame}(\llbracket E \rrbracket) \cup \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket E \rrbracket) \cup \mathit{footprint}(\llbracket D \rrbracket)), \\
\mathit{frame}(\llbracket E \rrbracket) \cup \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket E \rrbracket) \cup \mathit{footprint}(\llbracket D \rrbracket)))'\mathit{assertion}, \\
\mathit{frame}(\llbracket E \rrbracket) \cup \mathit{frame}(\llbracket D \rrbracket), \mathit{footprint}(\llbracket E \rrbracket) \cup \mathit{footprint}(\llbracket D \rrbracket)))
\end{array}\ \textsc{expand}
$$

Table 2 illustrates the list of proof rules that we use to prove the theorems of this section. We use the same proof strategy as with the list of lemmas of Appendix A: apply a series of proof rules until the two expressions of partial contracts under comparison are syntactically equal.

Similarly to the lemmas exposed in Appendix A, since the union operator already satisfies the algebraic properties that we aim to prove, we can avoid to reason about the equality between the frame and footprint resulting from the

Table 2. List of Proof Rules.

equivalent	Reformulate Orc expression using operator equivalence
rewrite	Rewrite Orc expression to new notation
$\oplus$ *comm*	Commutativity of $\oplus$, where $\oplus$ is either $\barwedge$, $\veebar$, $\triangle$ or $\blacktriangle$
$\oplus$ *assoc*	Associativity of $\oplus$, where $\oplus$ is either $\triangle$ or $\blacktriangle$
$\oplus$ *dist*	Distributivity of $\oplus$, where $\oplus$ is either $\triangle$ or $\blacktriangle$
dist $\oplus/\oslash$	Distributivity of either $\triangle$ over $\barwedge$ or $\blacktriangle$ over $\veebar$
$\veebar$ *idemp*	Idempotence of $\veebar$
pre swap	Precondition partial contract swap (Lemma 13)
post swap	Postcondition partial contract swap (Lemma 14)

semantic derivation of the two expressions under comparison. Thus, we only reason about the equality between the contracts of each Orc expression. Moreover, we analyze the precondition and postcondition equality separately, as each of these types of assertions capture different concerns.

Theorems for the Semantic Derivation Function

Theorem 1 (Commutativity of Parallel Operator). *Let C and D be any two Orc expressions. Since the Orc parallel operator is **commutative** i.e.,*

$$C \mid D = D \mid C$$

the semantic derivation of both expressions under comparison must yield the same result

$$[\![C \mid D]\!] = [\![D \mid C]\!]$$

Proof.

Precondition Derivation

$$\dfrac{\dfrac{[\![C \mid D]\!]_P = [\![D \mid C]\!]_P}{(\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P) = (\mathcal{D}_P \triangle \mathcal{C}_P) \barwedge (\mathcal{C}_P \triangle \mathcal{D}_P)}\ \textsc{rewrite}}{(\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P) = (\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P)}\ \barwedge\ \textsc{comm}$$

Postcondition Derivation

$$\dfrac{\dfrac{[\![C \mid D]\!]_Q = [\![D \mid C]\!]_Q}{(\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) = (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \veebar (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)}\ \textsc{rewrite}}{(\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) = (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)}\ \veebar\ \textsc{comm}$$

Theorem 2 (Associativity of Parallel Operator). *Let C, D, and E be any three Orc expressions. Since the Orc parallel operator is **associative** i.e.,*

$$(C \mid D) \mid E = C \mid (D \mid E)$$

the semantic derivation of both expressions under comparison must yield the same result

$$\llbracket (C \mid D) \mid E \rrbracket = \llbracket C \mid (D \mid E) \rrbracket$$

if C, D, and E are independent and non-interfering, or in the case that any of these expressions interferes with another expression, both expressions have the same frame and footprint, and the postcondition clauses mentioning any communication location of both expressions are identical.

Proof.

Precondition Derivation

$$\begin{array}{cl}
\llbracket (C \mid D) \mid E \rrbracket_P = \llbracket C \mid (D \mid E) \rrbracket_P & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P)) \triangle \mathcal{E}_P \barwedge \mathcal{E}_P \triangle ((\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \textsc{rewrite} \\
\mathcal{C}_P \triangle ((\mathcal{D}_P \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle \mathcal{D}_P)) \barwedge ((\mathcal{D}_P \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle \mathcal{D}_P)) \triangle \mathcal{C}_P & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \textsc{dist } \triangle/\barwedge \\
(\mathcal{C}_P \triangle (\mathcal{D}_P \triangle \mathcal{E}_P)) \barwedge (\mathcal{C}_P \triangle (\mathcal{E}_P \triangle \mathcal{D}_P)) \barwedge ((\mathcal{D}_P \triangle \mathcal{E}_P) \triangle \mathcal{C}_P) \barwedge ((\mathcal{E}_P \triangle \mathcal{D}_P) \triangle \mathcal{C}_P) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{C}_P \triangle \mathcal{E}_P)) \barwedge ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \triangle \textsc{ assoc} \\
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{C}_P \triangle \mathcal{E}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{C}_P \triangle \mathcal{E}_P)) \barwedge ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \triangle \textsc{ comm} \\
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{C}_P \triangle \mathcal{E}_P)) \barwedge ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) &
\end{array}$$

Postcondition Derivation

$$\begin{array}{cl}
\llbracket (C \mid D) \mid E \rrbracket_Q = \llbracket C \mid (D \mid E) \rrbracket_Q & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}'_Q)) \blacktriangle \mathcal{E}_Q \veebar \mathcal{E}'_Q \blacktriangle ((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \textsc{rewrite} \\
\mathcal{C}'_Q \blacktriangle ((\mathcal{D}'_Q \blacktriangle \mathcal{E}_Q) \veebar (\mathcal{E}'_Q \blacktriangle \mathcal{D}_Q)) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle \mathcal{D}'_Q)) \blacktriangle \mathcal{C}_Q & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{E}_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \textsc{dist } \blacktriangle/\veebar \\
(\mathcal{C}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{E}_Q)) \veebar (\mathcal{C}'_Q \blacktriangle (\mathcal{E}'_Q \blacktriangle \mathcal{D}_Q)) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{E}'_Q) \blacktriangle \mathcal{C}_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{C}_Q) & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{E}_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \blacktriangle \textsc{ comm} \\
(\mathcal{C}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{E}_Q)) \veebar (\mathcal{C}'_Q \blacktriangle (\mathcal{E}'_Q \blacktriangle \mathcal{D}_Q)) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{C}_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{C}_Q) & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \blacktriangle \textsc{ assoc} \\
((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{C}'_Q \blacktriangle \mathcal{E}'_Q) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \blacktriangle \textsc{ comm} \\
((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \veebar \textsc{ idemp} \\
((\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) &
\end{array}$$

Theorem 3 (Associativity of Sequential Operator). *Let C, D, and E be any three Orc expressions. Since the Orc sequential operator is* ***associative*** *i.e.,*

$$(C \gg D) \gg E = C \gg (D \gg E)$$

the semantic derivation of both expressions under comparison must yield the same result

$$\llbracket (C \gg D) \gg E \rrbracket = \llbracket C \gg (D \gg E) \rrbracket$$

By definition [2], $>x>$ is right associative*.*

Proof.

$$\textsc{Precondition Derivation}$$

$$\dfrac{\dfrac{[\![(C \gg D) \gg E]\!]_P = [\![C \gg (D \gg E)]\!]_P}{(\mathcal{C}_P \triangle \mathcal{D}_P)\triangle\mathcal{E}_P = \mathcal{C}_P\triangle(\mathcal{D}_P\triangle\mathcal{E}_P)}\ \textsc{rewrite}}{(\mathcal{C}_P \triangle \mathcal{D}_P)\triangle\mathcal{E}_P = (\mathcal{C}_P\triangle\mathcal{D}_P)\triangle\mathcal{E}_P}\ \triangle\ \textsc{assoc}$$

$$\textsc{Postcondition Derivation}$$

$$\dfrac{\dfrac{[\![(C \gg D) \gg E]\!]_Q = [\![C \gg (D \gg E)]\!]_Q}{(\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q)\blacktriangle\mathcal{E}_Q = \mathcal{C}'_Q\blacktriangle(\mathcal{D}'_Q\blacktriangle\mathcal{E}_Q)}\ \textsc{rewrite}}{(\mathcal{C}'_Q \blacktriangle \mathcal{D}'_Q)\blacktriangle\mathcal{E}_Q = (\mathcal{C}'_Q\blacktriangle\mathcal{D}'_Q)\blacktriangle\mathcal{E}_Q}\ \blacktriangle\ \textsc{assoc}$$

Theorem 4 (Right Distributivity of Sequential Operator over Parallel Operator). *Let C, D, and E be any three Orc expressions. Since the Orc sequential operator is **right distributive** over the parallel operator i.e.,*

$$(C \mid D) \gg E = (C \gg E) \mid (D \gg E)$$

the semantic derivation of both expressions under comparison must yield the same result

$$[\![(C \mid D) \gg E]\!] = [\![(C \gg E) \mid (D \gg E)]\!]$$

if C, D, and E are independent from E.

Proof.

$$\textsc{Precondition Derivation}$$

$$\dfrac{\dfrac{\dfrac{[\![(C \mid D) \gg E]\!]_P = [\![(C \gg E) \mid (D \gg E)]\!]_P}{\begin{array}{c}((\mathcal{C}_P\triangle\mathcal{D}_P)\barwedge(\mathcal{D}_P\triangle\mathcal{C}_P))\triangle\mathcal{E}_P = \\ ((\mathcal{C}_P\triangle\mathcal{E}_P)\triangle(\mathcal{D}_P\triangle\mathcal{E}_P))\barwedge((\mathcal{D}_P\triangle\mathcal{E}_P)\triangle(\mathcal{C}_P\triangle\mathcal{E}_P))\end{array}}\ \textsc{rewrite}}{\begin{array}{c}((\mathcal{C}_P\triangle\mathcal{D}_P)\triangle\mathcal{E}_P)\barwedge((\mathcal{D}_P\triangle\mathcal{C}_P)\triangle\mathcal{E}_P) = \\ ((\mathcal{C}_P\triangle\mathcal{E}_P)\triangle(\mathcal{D}_P\triangle\mathcal{E}_P))\barwedge((\mathcal{D}_P\triangle\mathcal{E}_P)\triangle(\mathcal{C}_P\triangle\mathcal{E}_P))\end{array}}\ \textsc{dist}\ \triangle/\barwedge}{\begin{array}{c}((\mathcal{C}_P\triangle\mathcal{D}_P)\triangle\mathcal{E}_P)\barwedge((\mathcal{D}_P\triangle\mathcal{C}_P)\triangle\mathcal{E}_P) = \\ ((\mathcal{C}_P\triangle\mathcal{D}_P)\triangle\mathcal{E}_P)\barwedge((\mathcal{D}_P\triangle\mathcal{C}_P)\triangle\mathcal{E}_P)\end{array}}\ \triangle\ \textsc{dist}$$

$$\textsc{Postcondition Derivation}$$

$$\dfrac{\dfrac{\dfrac{[\![(C \mid D) \gg E]\!]_Q = [\![(C \gg E) \mid (D \gg E)]\!]_Q}{\begin{array}{c}((\mathcal{C}'_Q\blacktriangle\mathcal{D}'_Q)\veebar(\mathcal{D}'_Q\blacktriangle\mathcal{C}'_Q))\blacktriangle\mathcal{E}_Q = \\ ((\mathcal{C}'_Q\blacktriangle\mathcal{E}'_Q)\blacktriangle(\mathcal{D}'_Q\blacktriangle\mathcal{E}_Q))\veebar((\mathcal{D}'_Q\blacktriangle\mathcal{E}'_Q)\blacktriangle(\mathcal{C}'_Q\blacktriangle\mathcal{E}_Q))\end{array}}\ \textsc{rewrite}}{\begin{array}{c}((\mathcal{C}'_Q\blacktriangle\mathcal{D}'_Q)\blacktriangle\mathcal{E}_Q)\veebar((\mathcal{D}'_Q\blacktriangle\mathcal{C}'_Q)\blacktriangle\mathcal{E}_Q) = \\ ((\mathcal{C}'_Q\blacktriangle\mathcal{E}'_Q)\blacktriangle(\mathcal{D}'_Q\blacktriangle\mathcal{E}_Q))\veebar((\mathcal{D}'_Q\blacktriangle\mathcal{E}'_Q)\blacktriangle(\mathcal{C}'_Q\blacktriangle\mathcal{E}_Q))\end{array}}\ \textsc{dist}\ \blacktriangle/\veebar}{\begin{array}{c}((\mathcal{C}'_Q\blacktriangle\mathcal{D}'_Q)\blacktriangle\mathcal{E}_Q)\veebar((\mathcal{D}'_Q\blacktriangle\mathcal{C}'_Q)\blacktriangle\mathcal{E}_Q) = \\ ((\mathcal{C}'_Q\blacktriangle\mathcal{D}'_Q)\blacktriangle\mathcal{E}_Q)\veebar((\mathcal{D}'_Q\blacktriangle\mathcal{C}'_Q)\blacktriangle\mathcal{E}_Q)\end{array}}\ \blacktriangle\ \textsc{dist}$$

Theorem 5 (Distributivity of Pruning Operator over Sequential Operator). *Let C, D, and E be any three Orc expressions. The Orc pruning operator is **distributive** over the sequential operator i.e.,*

$$(C \gg D) \ll E = (C \ll E) \gg D$$

if D and E are independent and non-interfering.

Thus, the semantic derivation of both expressions under comparison must yield the same result:

$$[\![(C \gg D) \ll E]\!] = [\![(C \ll E) \gg D]\!]$$

Proof. Consider the following two cases:

1. Expression E communicates with C via message passing

$$\cfrac{\cfrac{[\![(C(x) \gg D)<x<E]\!] = [\![(C(x)<x<E) \gg D]\!]}{[\![E>x>(C(x) \gg D)]\!] = [\![(E>x>C(x)) \gg D]\!]}\ \textsc{equivalent}}{[\![E>x>(C(x) \gg D)]\!] = [\![E>x>(C(x) \gg D)]\!]}\ \gg \textsc{assoc}$$

2. Expression E does not communicate with C via message passing

Precondition Derivation

$$\cfrac{\cfrac{\cfrac{[\![(C \gg D) \ll E]\!]_P = [\![(C \ll E) \gg D]\!]_P}{((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) = ((\mathcal{C}_P \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle \mathcal{C}_P)) \triangle \mathcal{D}_P}\ \textsc{rewrite}}{((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) = ((\mathcal{C}_P \triangle \mathcal{E}_P) \triangle \mathcal{D}_P) \barwedge ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P)}\ \textsc{dist}\ \triangle/\barwedge}{((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) = ((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P))}\ \textsc{pre swap}$$

Postcondition Derivation

$$\cfrac{\cfrac{\cfrac{[\![(C \gg D) \ll E]\!]_Q = [\![(C \ll E) \gg D]\!]_Q}{((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) = ((\mathcal{C}'_Q \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle \mathcal{C}'_Q)) \blacktriangle \mathcal{D}_Q}\ \textsc{rewrite}}{((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) = ((\mathcal{C}'_Q \blacktriangle \mathcal{E}'_Q) \blacktriangle \mathcal{D}_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{D}_Q)}\ \textsc{dist}\ \blacktriangle/\veebar}{((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) = ((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q))}\ \textsc{post swap}$$

Theorem 6 (Distributivity of Pruning Operator over Parallel Operator). *Let C, D, and E be any three Orc expressions. The Orc pruning operator is **distributive** over the parallel operator i.e.,*

$$(C \mid D) \ll E = (C \ll E) \mid D$$

if D and E are independent and non-interfering.

Thus, the semantic derivation of both expressions under comparison must yield the same result:

$$[\![(C \mid D) \ll E]\!] = [\![(C \ll E) \mid D]\!]$$

Proof. Consider the following two cases:

1. Expression E communicates with C via message passing

PRECONDITION DERIVATION

$$
\begin{array}{cl}
\llbracket (C(x) \mid D) {<} x {<} E \rrbracket_P = \llbracket (C(x) {<} x {<} E) \mid D \rrbracket_P & \\ \hline
\mathcal{E}_P \triangle ((\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P)) = ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \text{REWRITE} \\ \hline
(\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \text{DIST } \triangle / \barwedge \\ \hline
(\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \triangle \text{ ASSOC} \\ \hline
(\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) & \text{PRE SWAP}
\end{array}
$$

POSTCONDITION DERIVATION

$$
\begin{array}{cl}
\llbracket (C(x) \mid D) {<} x {<} E \rrbracket_Q = \llbracket (C(x) {<} x {<} E) \mid D \rrbracket_Q & \\ \hline
\mathcal{E}''_Q \blacktriangle ((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = ((\mathcal{E}''_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}''_Q \blacktriangle \mathcal{C}_Q)) & \text{REWRITE} \\ \hline
(\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}''_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = ((\mathcal{E}''_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}''_Q \blacktriangle \mathcal{C}_Q)) & \text{DIST } \blacktriangle / \veebar \\ \hline
(\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}''_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = (\mathcal{E}''_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}''_Q \blacktriangle \mathcal{C}_Q)) & \blacktriangle \text{ ASSOC} \\ \hline
(\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}''_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}''_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) & \text{POST SWAP}
\end{array}
$$

2. Expression E does not communicate with C via message passing

PRECONDITION DERIVATION

$$
\begin{array}{cl}
\llbracket (C \mid D) \ll E \rrbracket_P = \llbracket (C \ll E) \mid D \rrbracket_P & \\ \hline
(((\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P)) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle ((\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P))) = & \text{REWRITE} \\
(((\mathcal{C}_P \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle \mathcal{C}_P)) \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle ((\mathcal{C}_P \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle \mathcal{C}_P))) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \text{DIST } \triangle / \barwedge \\
((\mathcal{C}_P \triangle \mathcal{E}_P) \triangle \mathcal{D}_P) \barwedge ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{C}_P \triangle \mathcal{E}_P)) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \triangle \text{ ASSOC} \\
((\mathcal{C}_P \triangle \mathcal{E}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \text{PRE SWAP} \\
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \barwedge \text{ COMM} \\
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) &
\end{array}
$$

POSTCONDITION DERIVATION

$$
\begin{array}{cl}
\llbracket (C \mid D) \ll E \rrbracket_Q = \llbracket (C \ll E) \mid D \rrbracket_Q & \\ \hline
(((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle ((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q))) = & \text{REWRITE} \\
(((\mathcal{C}'_Q \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle \mathcal{C}'_Q)) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle ((\mathcal{C}_Q \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q))) & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \text{DIST } \blacktriangle / \veebar \\
((\mathcal{C}'_Q \blacktriangle \mathcal{E}'_Q) \blacktriangle \mathcal{D}_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{C}'_Q) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{E}'_Q)) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q)) & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \blacktriangle \text{ ASSOC} \\
((\mathcal{C}'_Q \blacktriangle \mathcal{E}'_Q) \blacktriangle \mathcal{D}_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q)) & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \text{POST SWAP} \\
((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) & \\ \hline
((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \veebar \text{ COMM} \\
((\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}'_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) &
\end{array}
$$

Theorem 7 (Distributivity of Pruning Operator over Pruning Operator). *Let C, D, and E be any three Orc expressions. The Orc pruning operator* *is **distributive** over itself i.e.,*

$$(C \ll D) \ll E = (C \ll E) \ll D$$

if D and E are independent and non-interfering.

Thus, the semantic derivation of both expressions under comparison must yield the same result:

$$[\![(C \ll D) \ll E]\!] = [\![(C \ll E) \ll D]\!]$$

Proof. Consider the following four cases:

1. *Expressions D and E communicate with C via message passing*

Precondition Derivation

$$
\begin{array}{c l}
[\![(C(x,y)<y<D)<x<E]\!]_P = [\![(C(x,y)<x<E)<y<D]\!]_P & \\ \hline
[\![E>x>(D>y>C(x,y))]\!]_P = [\![D>y>(E>x>C(x,y))]\!]_P & \textsc{equivalent} \\ \hline
\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P) = \mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P) & \textsc{rewrite} \\ \hline
\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P) = \mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P) & \textsc{pre swap}
\end{array}
$$

Postcondition Derivation

$$
\begin{array}{c l}
[\![(C(x,y)<y<D)<x<E]\!]_Q = [\![(C(x,y)<x<E)<y<D]\!]_Q & \\ \hline
[\![E>x>(D>y>C(x,y))]\!]_Q = [\![D>y>(E>x>C(x,y))]\!]_Q & \textsc{equivalent} \\ \hline
\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) = \mathcal{D}'_Q \blacktriangle (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q) & \textsc{rewrite} \\ \hline
\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) = \mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) & \textsc{post swap}
\end{array}
$$

2. *Only expression D communicates with C via message passing*

Precondition Derivation

$$
\begin{array}{c l}
[\![(C(x)<x<D) \ll E]\!]_P = [\![(C(x) \ll E)<x<D]\!]_P & \\ \hline
((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = \mathcal{D}_P \triangle ((\mathcal{C}_P \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle \mathcal{C}_P)) & \textsc{rewrite} \\ \hline
((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = (\mathcal{D}_P \triangle (\mathcal{C}_P \triangle \mathcal{E}_P)) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \textsc{dist } \triangle / \barwedge \\ \hline
((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \triangle \textsc{ assoc} \\ \hline
((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) & \textsc{pre swap}
\end{array}
$$

Postcondition Derivation

$$
\begin{array}{c l}
[\![(C(x)<x<D) \ll E]\!]_Q = [\![(C(x) \ll E)<x<D]\!]_Q & \\ \hline
((\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q)) = \mathcal{D}''_Q \blacktriangle ((\mathcal{C}_Q \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q)) & \textsc{rewrite} \\ \hline
((\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q)) = (\mathcal{D}''_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{E}'_Q)) \veebar (\mathcal{D}''_Q \blacktriangle (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q)) & \textsc{dist } \blacktriangle / \veebar \\ \hline
((\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q)) = ((\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{D}''_Q \blacktriangle (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q)) & \blacktriangle \textsc{ assoc} \\ \hline
((\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q)) = ((\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}''_Q \blacktriangle \mathcal{C}_Q)) & \textsc{post swap}
\end{array}
$$

3. Only expression E communicates with C via message passing

Precondition Derivation

$$\begin{array}{cl}
\llbracket (C(x) \ll D){<}x{<}E \rrbracket_P = \llbracket (C(x){<}x{<}E) \ll D \rrbracket_P & \\ \hline
\mathcal{E}_P \triangle ((\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P)) = ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \text{REWRITE} \\ \hline
(\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \text{DIST } \triangle/\barwedge \\ \hline
(\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \triangle \text{ ASSOC} \\ \hline
(\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) & \text{PRE SWAP}
\end{array}$$

Postcondition Derivation

$$\begin{array}{cl}
\llbracket (C(x) \ll D){<}x{<}E \rrbracket_Q = \llbracket (C(x){<}x{<}E) \ll D \rrbracket_Q & \\ \hline
\mathcal{E}''_Q \blacktriangle ((\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = ((\mathcal{E}''_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}''_Q \blacktriangle \mathcal{C}_Q)) & \text{REWRITE} \\ \hline
(\mathcal{E}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{E}''_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = ((\mathcal{E}''_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}''_Q \blacktriangle \mathcal{C}_Q)) & \text{DIST } \blacktriangle/\veebar \\ \hline
(\mathcal{E}''_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{E}''_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = (\mathcal{E}''_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}''_Q \blacktriangle \mathcal{C}_Q)) & \blacktriangle \text{ ASSOC} \\ \hline
(\mathcal{E}''_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{E}''_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = (\mathcal{E}''_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{E}''_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) & \text{POST SWAP}
\end{array}$$

4. Neither expression D nor E communicates with C via message passing

Precondition Derivation

$$\begin{array}{cl}
\llbracket (C \ll D) \ll E \rrbracket_P = \llbracket (C \ll E) \ll D \rrbracket_P & \\ \hline
(((\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P)) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle ((\mathcal{C}_P \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle \mathcal{C}_P))) = & \text{REWRITE} \\
(((\mathcal{C}_P \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle \mathcal{C}_P)) \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle ((\mathcal{C}_P \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle \mathcal{C}_P))) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \text{DIST } \triangle/\barwedge \\
((\mathcal{C}_P \triangle \mathcal{E}_P) \triangle \mathcal{D}_P) \barwedge ((\mathcal{E}_P \triangle \mathcal{C}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{C}_P \triangle \mathcal{E}_P)) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \triangle \text{ ASSOC} \\
((\mathcal{C}_P \triangle \mathcal{E}_P) \triangle \mathcal{D}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{D}_P \triangle (\mathcal{E}_P \triangle \mathcal{C}_P)) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \text{PRE SWAP} \\
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) & \\ \hline
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) = & \barwedge \text{ COMM} \\
((\mathcal{C}_P \triangle \mathcal{D}_P) \triangle \mathcal{E}_P) \barwedge ((\mathcal{D}_P \triangle \mathcal{C}_P) \triangle \mathcal{E}_P) \barwedge (\mathcal{E}_P \triangle (\mathcal{C}_P \triangle \mathcal{D}_P)) \barwedge (\mathcal{E}_P \triangle (\mathcal{D}_P \triangle \mathcal{C}_P)) &
\end{array}$$

Postcondition Derivation

$$\begin{array}{cl}
\llbracket (C \ll D) \ll E \rrbracket_Q = \llbracket (C \ll E) \ll D \rrbracket_Q & \\ \hline
(((\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle ((\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q))) = & \text{REWRITE} \\
(((\mathcal{C}_Q \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q)) \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle ((\mathcal{C}_Q \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q))) & \\ \hline
((\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \text{DIST } \blacktriangle/\veebar \\
((\mathcal{C}_Q \blacktriangle \mathcal{E}'_Q) \blacktriangle \mathcal{D}'_Q) \veebar ((\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{E}'_Q)) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q)) & \\ \hline
((\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \blacktriangle \text{ ASSOC} \\
((\mathcal{C}_Q \blacktriangle \mathcal{E}'_Q) \blacktriangle \mathcal{D}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{D}'_Q \blacktriangle (\mathcal{E}'_Q \blacktriangle \mathcal{C}_Q)) & \\ \hline
((\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \text{POST SWAP} \\
((\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) & \\ \hline
((\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) = & \veebar \text{ COMM} \\
((\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q) \blacktriangle \mathcal{E}'_Q) \veebar ((\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q) \blacktriangle \mathcal{E}'_Q) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{C}_Q \blacktriangle \mathcal{D}'_Q)) \veebar (\mathcal{E}'_Q \blacktriangle (\mathcal{D}'_Q \blacktriangle \mathcal{C}_Q)) &
\end{array}$$

References

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